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FUNCTIONS OF THREE REAL VARIABLES

(Reprint from American Journal of Mathematics)

THESIS

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN THE GRADUATE SCHOOL
IN THE UNIVERSITY OF ILLINOIS

1903

BY

HENRY LIVINGSTON COAR

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Functions of Three Real Independent Variables.

BY HENRY LIVINGSTON COAR.

CHAPTER I.

SIMULTANEOUS LIMITS.

1. *Definitions.*—Let $u = f(x, y, z)$ be a function of the three real independent variables x, y, and z, and let u be completely defined and single valued for all values of the variables, that is to say, u shall have one and only one definite value for each set of values of the variables x, y, and z. If (x_0, y_0, z_0) is such a set of values of the variables, then these may represent the coordinates of a point in space of three dimensions. For such values (x_0, y_0, z_0) the function $f(x, y, z)$ will have a definite value which we shall denote by $f(x_0, y_0, z_0)$. It will be convenient to call $f(x_0, y_0, z_0)$ "the value of the function at the point (x_0, y_0, z_0)".

We must now define limiting values and limits. We say that x_0 is a limiting value of the variable x, if there exist values of x in every neighborhood of x_0. This may also be expressed by saying that the values of x are dense at x_0. Similarly, if x, y, and z are dense, respectively, at x_0, y_0, and z_0, then (x_0, y_0, z_0) is a limiting value of (x, y, z). In this case (x, y, z) may approach (x_0, y_0, z_0) in any one of three different ways, or, to use the language of geometry, the point (x, y, z) may approach the point (x_0, y_0, z_0) in any one of three different ways. First, we may let each of the variables vary by itself, that is, we may regard any two of the variables as constant and let the third vary. Secondly, we may let two of the variables vary simultaneously while we regard the third variable as constant. Lastly, we may let all three variables vary simultaneously. These three different kinds of variation of the variables give rise to three different kinds of limits.

Before defining these let us recall briefly the different kinds of limits we may have in the case of a function of two real variables, namely, single limits, double limits, and simultaneous limits. These are defined as follows, x_0 and y_0

being in every case the limiting values of x and y, respectively. We say that the single limit $\lim\limits_{x=x_0} f(x, y)$ exists and equals A, if having chosen an arbitrarily small $\sigma > 0$, we can then find a $\delta > 0$, such that $|f(x_0+\delta, y_0) - A| < \sigma$. We say that the double limit $\lim\limits_{x=x_0} \lim\limits_{y=y_0} f(x, y)$ exists if each of the following limits exists: $\lim\limits_{y=y_0} f(x, y) = F(x, y_0)$ and $\lim\limits_{x=x_0} F(x, y_0) = F(x_0, y_0)$. Lastly we say that the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0}} f(x, y)$ exists and equals A, if having chosen an arbitrarily small $\sigma > 0$, we can then find a $\delta > 0$, such that $|f(x_0+\delta_1, y_0+\delta_2) - A| < \sigma$, for every pair of values (δ_1, δ_2), where δ_1 and δ_2 are independent of each other and satisfy simultaneously the inequalities $|\delta_1| =< \delta$, $|\delta_2| =< \delta$.

We shall now define the different kinds of limits for a function of three variables. In what follows the values x, y, and z are supposed to be dense at x_0, y_0, and z_0, respectively.

(1) We say that the limit $\lim\limits_{x=x_0} \lim\limits_{y=y_0} \lim\limits_{z=z_0} f(x, y, z)$ exists if the following single limits exist: $\lim\limits_{z=z_0} f(x, y, z) = F(x, y, z_0)$, $\lim\limits_{y=y_0} F(x, y, z_0) = F_1(x, y_0, z_0)$, and $\lim\limits_{z=z_0} F_1(x, y_0, z_0) = F_2(x_0, y_0, z_0)$. We call the limit $\lim\limits_{x=x_0} \lim\limits_{y=y_0} \lim\limits_{z=z_0} f(x, y, z)$ the *triple limit.*

(2) We say that the limit $\lim\limits_{\substack{x=x_0\\ y=y_0}} \lim\limits_{z=z_0} f(x, y, z)$ exists, if the single limit $\lim\limits_{z=z_0} f(x, y, z) = F(x, y, z_0)$ and the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0}} F(x, y, z_0) = F_1(x_0, y_0, z_0)$, both exist. The limit $\lim\limits_{z=z_0} \lim\limits_{\substack{x=x_0\\ y=y_0}} f(x, y, z)$ is defined in an analogous manner. Each of these limits is called a *double simultaneous limit.*

(3) Finally we say that the limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals A, if having chosen at pleasure a $\sigma > 0$, we can then find a $\delta > 0$, such that

$$| f(x_0+\delta_1,\ y_0+\delta_2,\ z_0+\delta_3) - A | < \sigma$$

where δ_1, δ_2, and δ_3, are independent of each other and satisfy simultaneously the inequalities $|\delta_1| \leqq \delta$, $|\delta_2| \leqq \delta$, $|\delta_3| \leqq \delta$. The limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ is called *the simultaneous limit in three variables.*

In any discussion of limits we must distinguish clearly between the value of a function at a point (x_0, y_0, z_0) and the limiting value of the function as (x, y, z) approaches (x_0, y_0, z_0). The former is dependent simply upon the value of (x_0, y_0, z_0), while the latter depends only upon the values of the function in the neighborhood of the point (x_0, y_0, z_0) and not upon the value of the function at (x_0, y_0, z_0), as is apparent from the following example.

Example.—Let $f(x, y, z) = x^2 + y^2 + z^2$ for all points excepting the origin, where the value of the function shall be unity. Then $\lim\limits_{\substack{x=0\\ y=0\\ z=0}} f(x, y, z) = 0$, while $f(0, 0, 0) = 1$.

2. *Properties of simultaneous limits.*—It is evident that we may let (x, y, z) approach (x_0, y_0, z_0) in an infinite number of different ways, and, moreover, that we may let it do so simultaneously. We must therefore meet the question as to whether we shall always obtain the same limiting value of the function if we let the variable approach the limiting value in different ways. The following examples show that different approaches may give rise to different values of the limiting function. In every case we suppose that the approach is a continuous one.

Example 1. Let $f(x, y, z) = \dfrac{x^2 - y^2 - z^2}{x^2 + y^2 + z^2}$.

Now let $x = 2t$, $y = 3t$, $z = 4t$, so that x, y, and z will approach zero simultaneously, if we let t approach zero. Then

$$f(x, y, z) = \frac{4t^2 - 9t^2 - 16t^2}{4t^2 + 9t^2 + 16t^2} = -\frac{21}{29},$$

so that in case of this approach $\lim\limits_{t=0} f(x, y, z) = -\frac{21}{29}$. Again let $x = t$, $y = -t$, and $z = 2t$, so that

$$f(x, y, z) = \frac{t^2 - t^2 - 4t^2}{t^2 + t^2 + 4t^2} = -\frac{2}{3},$$

so that $\lim\limits_{t=0} f(x, y, z) = -\frac{2}{3}$.

We see from this example that for the two different linear approaches in question we obtain two different values of the limiting function. In this case we can therefore evidently not say that the simultaneous limit $\lim\limits_{\substack{x=0\\y=0\\z=0}} f(x, y, z)$ exists. But even if we should obtain the same limiting value of the function for every simultaneous linear approach of (x, y, z) to (x_0, y_0, z_0), we cannot say that the simultaneous limit will exist, for it may be possible that we shall obtain other limiting values for approaches of higher order. This is illustrated by the following example.

Example 2. Let $f(x, y, z) = \frac{xy + z}{xy - z}$.

Now let $x = at$, $y = bt$, $z = ct$, where a, b, and c may have any values whatever, so that this will give us any linear approach. Then we have

$$f(x, y, z) = \frac{abt^2 + ct}{abt^2 - ct} = \frac{abt + c}{abt - c},$$

so that $\lim\limits_{t=0} f(x, y, z) = -1$, which is the limiting value for every linear approach.

Now let $x = t$, $y = t$, $z = t^2$, so that

$$f(x, y, z) = \frac{t^2 + t^2}{t^2 - t^2} = \infty,$$

so that this quadratic approach gives us an infinite value of the function.

The reason why we obtain different limiting values of the function in the above examples is to be found in the fact that the limiting value of the function depends only upon the values of the function in the neighborhood of the point $(0, 0, 0)$ and not upon the value of the function at the point $(0, 0, 0)$. We are now in a position to state and prove the following theorem.

Theorem 1.—The necessary and sufficient condition that the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ shall exist is that we always obtain the same definite limiting value A of the function, no matter in what manner (x, y, z) approaches (x_0, y_0, z_0) simultaneously and continuously.

The condition is necessary, that is to say, if the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals A; then, no matter how x, y, and z approach (x_0, y_0, z_0) simultaneously and continuously, we always obtain the same limiting value of the function, namely A. Let $x=\phi(t)$, $y=\psi(t)$, and $z=\chi(t)$, where ϕ, ψ, and χ are continuous functions of t. Then as t approaches t_0, we have in the limit

$$\lim_{t=t_0} \phi(t)=\phi(t_0)=x_0, \ \lim_{t=t_0} \psi(t)=\psi(t_0)=y_0, \ \lim_{t=t_0} \chi(t)=\chi(t_0)=z_0.$$

Making these substitutions in the original function, we obtain

$$f(x, y, z)=f[\phi(t), \psi(t), \chi(t)].$$

Our conclusion asserts that $\lim\limits_{t=t_0} f[\phi(t), \psi(t), \chi(t)]=A$.

Suppose that this is not so. In that case we can choose at pleasure a $\sigma > 0$ and can then determine a $\bar{\delta} > 0$, such that

$$|f[\psi(t_0+\delta), \psi(t_0+\delta), \chi(t_0+\delta)-A| > \sigma, \tag{1}$$

where $\delta < \bar{\delta}$. But since ϕ, ψ, and χ are continuous functions of t, we have

$$\begin{aligned}\phi(t_0+\delta) &= \phi(t_0)+\delta_1 = x_0+\delta_1,\\ \psi(t_0+\delta) &= \psi(t_0)+\delta_2 = y_0+\delta_2,\\ \chi(t_0+\delta) &= \chi(t_0)+\delta_3 = z_0+\delta_3,\end{aligned}$$

where $|\delta_1|, |\delta_2|$, and $|\delta_3|$ can be made as small as we please by making δ sufficiently small. We can therefore make each of the δ's less than $\bar{\delta}$ which we do. Substituting these values in (1), we obtain

$$|f(x_0+\delta_1, y_0+\delta_2, z_0+\delta_3)-A| > \sigma.$$

This contradicts our definition of a simultaneous limit, since $|\delta_1| \leqq \bar{\delta}, |\delta_2| \leqq \bar{\delta}$, $|\delta_3| \leqq \bar{\delta}$. Hence the condition is necessary.

The condition is also sufficient. In this case we have

$$\lim_{t=t_0} f[\phi(t), \psi(t), \chi(t)] = A,$$

and we must prove that

$$|f(x_0+\delta_1, y_0+\delta_2, z_0+\delta_3) - A| < \sigma.$$

If this is not true we shall have

$$|f(x_0+\delta_1, y_0+\delta_2, z_0+\delta_3) - A| \geqq \sigma,$$

and hence

$$|f[\psi(t_0+\delta_1), \psi(t_0+\delta_2), \chi(t_0+\delta_3) - A| \geqq \sigma,$$

which contradicts the hypothesis.

It was not necessary to express everything in terms of the new parameter t. The substitution $z = F(x, y)$, continuous in (x, y) together, and $y = F_1(x)$, continuous in x, would have let us express everything in terms of x, and the nature of the proof would remain unchanged.

The theorem just proved is especially useful in enabling us to decide when the simultaneous limit does not exist, as is shown by examples 1 and 2 of this section.

Theorem 2.—If the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals A, then we must also have:

(*a*) $\lim\limits_{\substack{x=x_0\\ y=y_0}} f(x, y, z_0) = \lim\limits_{\substack{x=x_0\\ z=z_0}} f(x, y_0, z) = \lim\limits_{\substack{y=y_0\\ z=z_0}} f(x_0, y, z) = A.$

(*b*) $\lim\limits_{x=x_0} f(x, y_0, z_0) = \lim\limits_{y=y_0} f(x_0, y, z_0) = \lim\limits_{z=z_0} f(x_0, y_0, z) = A.$

Case (*a*). Let $z = F(x, y) = z_0$. Then, since the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists, we have (by theorem 1), when we select the definite approach $z = F(x, y) = z_0$,

$$|f[x, y, F(x, y)] - A| < \sigma$$

or

$$|f(x, y, z_0) - A| < \sigma,$$

which proves the truth of the statement. In a like manner we prove the theorem for the other limits.

Case (*b*). Let $y=\psi(t)$ and $z=\chi(t)$, where ψ and χ are continuous functions of t and where we also have

$$\lim_{t=t_0}\psi(t)=\psi(t_0)=y_0,\ \lim_{t=y_0}\chi(t)=\chi(t_0)=z_0.$$

This, therefore, gives us a definite manner of approach, and since the simultaneous limit exists by hypothesis, we have (by virtue of theorem 1) for this approach

$$|\,f[x,\psi(t_0),\chi(t_0)]-A\,|<\sigma,$$

or

$$|\,f(x,y_0,z_0)-A\,|<\sigma.$$

This proves the theorem. The proof for the other limits is similar.

The converse of theorem 2 is, however, not true, as is evident from the following examples.

Example 3. Let $f(x,y,z)=\dfrac{xyz}{x^3+y^3+z^3}.$

In this example we have

$$\lim_{\substack{x=0\\y=0}} f(x,y,0)=\lim_{\substack{x=0\\z=0}} f(x,0,z)=\lim_{\substack{y=0\\z=0}} f(0,y,z)=0,$$

but the simultaneous limit $\lim\limits_{\substack{x=0\\y=0\\z=0}} f(x,y,z)$ does not exist; for let $y=z=mx$. Then we have

$$\lim_{\substack{x=0\\y=0\\z=0}}\frac{xyz}{x^3+y^3+z^3}=\lim_{x=0}\frac{m^2x^3}{x^3+2m^3x^3}=\frac{m^2}{1+2m^3}$$

which may assume all values between 0 and 1/3, as m varies from 0 to 1.

Example 4. Let $f(x,y,z)=\dfrac{xyz^3}{x^3+y^3+z^6}.$

In this example we have

$$\lim_{x=0} f(x,0,0)=\lim_{y=0} f(0,y,0)=\lim_{z=0} f(0,0,z)=0,$$

but the simultaneous limit $\lim\limits_{\substack{x=0\\y=0\\z=0}} f(x,y,z)$ does not exist; for let $y=z=mx$.

Then we have

$$\lim_{\substack{x=0\\y=0\\z=0}} \frac{xyz^3}{x^8+y^8+z^6}=\lim_{x=0} \frac{m^8x^4}{(1+m^8)x^8+m^6x^6}=0$$

for this approach. Again, let $y=z=x^2$. This approach gives us

$$\lim_{\substack{x=0\\y=0\\z=0}} \frac{xyz^3}{x^8+y^8+z^6}=\lim_{x=0} \frac{x^8}{x^8+x^8+x^8}=1/3.$$

The simultaneous limit, therefore, does not exist by virtue of theorem 1. In this example we obtain the same limiting value for every linear approach, but not for every other approach. This emphasizes the importance of examining all possible approaches when we wish to see if the simultaneous limit exists.

In theorem 2 we have shown that from the existence of the simultaneous limit in three variables follows the existence of certain single limits and also of certain simultaneous limits in two variables. We might, therefore, infer that in this case the following limits also exist:

(*a*) $$\lim_{x=x_0}\lim_{y=y_0} f(x,y,z_0)=\lim_{y=y_0}\lim_{x=x_0} f(x,y,z_0)=\lim_{x=x_0}\lim_{z=z_0} f(x,y_0,z)=$$

$$\lim_{z=z_0}\lim_{x=x_0} f(x,y_0,z)=\lim_{y=y_0}\lim_{z=z_0} f(x_0,y,z)=\lim_{z=z_0}\lim_{y=y_0} f(x_0,y,z)=A.$$

(*b*) $$\lim_{x=x_0}\lim_{y=y_0}\lim_{z=z_0} f(x,y,z)=\lim_{x=x_0}\lim_{z=z_0}\lim_{y=y_0} f(x,y,z)=\text{etc.}=A.$$

(*c*) $$\lim_{\substack{x=x_0\\y=y_0}} f(x,y,z_0+\varepsilon_1)=\lim_{\substack{x=x_0\\z=z_0}} f(x,y_0+\varepsilon_2,z)=\lim_{\substack{y=y_0\\z=z_0}} f(x_0+\varepsilon_3,y,z),$$ where ε_1, ε_2

and ε_3 may be arbitrarily small. This inference is, however, not warranted, as is shown by the following examples.

Example 5. Let $f(x,y,z)=z\sin 1/xy$.

In this case we always obtain the same limiting value zero at the origin, no matter in what manner x, y, and z approach simultaneously to $(0, 0, 0)$; hence the simultaneous limit $\lim\limits_{\substack{x=0\\ y=0\\ z=0}} f(x, y, z)$ exists, by theorem 1. It follows from theorem 2 that

$$\lim_{\substack{x=0\\ y=0}} f(x, y, 0) = \lim_{\substack{x=0\\ z=0}} f(x, 0, z) = \lim_{\substack{y=0\\ z=0}} f(0, y, z) = 0,$$

and also that

$$\lim_{x=0} f(x, 0, 0) = \lim_{y=0} f(0, y, 0) = \lim_{z=0} f(0, 0, z) = 0.$$

But the limits under (*a*) do not exist, for $\lim\limits_{x=0}\lim\limits_{y=0} z \sin 1/xy$ has no meaning.

The same example also shows that the limits under (*b*) need not exist, for $\lim\limits_{z=0}\lim\limits_{x=0}\lim\limits_{y=0} z \sin 1/xy$ has no meaning. The existence of this limit would namely imply the existence of the limit $\lim\limits_{x=0}\lim\limits_{y=0} z \sin 1/xy$. This, however, does not exist, for if we let $z=2$, then the limit $\lim\limits_{x=0}\lim\limits_{y=0} z \sin \frac{1}{xy}$ may assume any value between $+2$ and -2. This example therefore shows that the existence of the simultaneous limit in three variables is not sufficient for the existence of the limits under (*a*) and (*b*). In the following example none of the limits under (*c*) exist.

Example 6. Let $f(x, y, z) = x \sin 1/y \sin 1/z$.

Here the simultaneous limit $\lim\limits_{\substack{x=0\\ y=0\\ z=0}} f(x, y, z)$ exists, but the limit $\lim\limits_{y=0}\lim\limits_{z=0} \varepsilon \sin 1/y \sin 1/z$ has no meaning, since it may assume all values between $+\varepsilon$ and $-\varepsilon$.

The converse of theorem 2 is also not true, that is, the simultaneous limits $\lim\limits_{\substack{x=x_0\\ y=y_0}} f(x, y, z)$, $\lim\limits_{\substack{y=y_0\\ z=z_0}} f(x, y, z)$, etc., may exist for every x, y, and z, respectively, in the neighborhood of (x_0, y_0, z_0), without necessitating the existence of the simultaneous limit in three variables. Likewise the single limits $\lim\limits_{x=x_0} f(x, y, z)$, etc., may exist for every pair of value (x, y), (x, z), (y, z), in the neighborhood

(x_0, y_0, z_0), while, nevertheless, the simultaneous limit in three variables does not exist. We show this by means of the following example.

Example 7. Let $f(x, y, z) = \dfrac{xyz}{x^3 + y^3 + z^3}$.

The following simultaneous limits exist:

$$\lim_{\substack{x=0\\ y=0}} f(x, y, z) = \lim_{\substack{x=0\\ y=0}} \frac{xyz}{x^3 + y^3 + z^3} = 0, \text{ when } z = 0,$$

$$\lim_{\substack{x=0\\ y=0}} f(x, y, z) = \lim_{\substack{x=0\\ y=0}} \frac{xyz}{x^3 + y^3} = 0, \text{ when } z \neq 0.$$

We see at once from the symmetry of the function that the other simultaneous limits in two variables also exist and equal zero, but nevertheless the simultaneous limit in three variables does not exist, as has already been shown. In the case of this example the single limits also exist and are all equal to each other, namely,

$$\lim_{x=0} f(x, y, z) = \lim_{x=0} \frac{0}{x^3} = 0, \text{ when } y = 0 \text{ and } z = 0,$$

$$\lim_{x=0} f(x, y, z) = \lim_{x=0} \frac{0}{x^3 + y^3} = 0, \text{ when } y = 0 \text{ and } z \neq 0,$$

$$\lim_{x=0} f(x, y, z) = \lim_{x=0} \frac{xyz}{x^3 + y^3 + z^3} = 0, \text{ when } y \neq 0 \text{ and } z \neq 0.$$

From the symmetry of the function we see that the other single limits also exist and equal zero, but the simultaneous limit in three variables does not exist.

In theorem 2 we have deduced the existence of certain limits from the existence of the simultaneous limit in three variables. The following theorem takes us a step further.

Theorem 3.—If the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals A, and if also

$$\lim_{\substack{x=x_0\\ y=y_0}} f(x,y,z) \text{ exists for every } z \text{ in the neighborhood of } z_0,$$

$$\lim_{\substack{x=x_0\\ z=z_0}} f(x,y,z) \quad \text{“} \quad \text{“} \quad y \quad \text{“} \quad \text{“} \quad y_0,$$

$$\lim_{\substack{y=y_0\\ z=z_0}} f(x,y,z) \quad \text{“} \quad \text{“} \quad x \quad \text{“} \quad \text{“} \quad x_0,$$

then we have

$$\lim_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x,y,z)=\lim_{x=x_0}\ \lim_{\substack{y=y_0\\ z=z_0}} f(x,y,z)=\lim_{y=y_0}\ \lim_{\substack{x=x_0\\ z=z_0}} f(x,y,z)=\lim_{z=z_0}\ \lim_{\substack{x=x_0\\ y=y_0}} f(x,y,z)=A.$$

Since by hypothesis the simultaneous limit in two variables exists, we have $\lim\limits_{\substack{x=x_0\\ y=y_0}} f(x,y,z) = F(x_0, y_0, z)$, where however $F(x_0, y_0, z)$ need not equal $f(x_0, y_0, z)$. Hence

$$|F(x_0, y_0, z) - f(x_0 + \delta_1, y_0 + \delta_2, z)| < \sigma, \tag{1}$$

for every z where $z_0 < z \leqq z_0 + \delta_3$ and $|\delta_1| \leqq |\delta|, |\delta_2| \leqq |\delta|, |\delta_3| \leqq |\delta|$ simultaneously. Since, also by hypothesis, the simultaneous limit in three variables exists and equals A, we have

$$|f(x_0 + \delta'_1, y_0 + \delta'_2, z) - A| < \sigma, \tag{2}$$

for every value of z where $z_0 < z \leqq z_0 + \delta'_3$, and $|\delta'_1| \leqq |\delta|, |\delta'_2| \leqq |\delta|, |\delta'_3| = |\delta|$ simultaneously. Now let $|\delta_1| \leqq |\delta'_1|, |\delta_2| \leqq |\delta'_2|, |\delta_3| \leqq |\delta'_3|$; then we have

$$|f(x_0 + \delta_1, y_0 + \delta_2, z) - A| < \sigma, \tag{3}$$

for every value of z, where $z_0 < z \leqq z_0 + \delta_3$. We may therefore add (1) and (3) and thus obtain

$$|F(x_0, y_0, z) - A| < 2\sigma,$$

for every value of z, where $z_0 < z \leqq z_0 + \delta_3$. Hence

$$\lim_{z=z_0} F(x_0, y_0, z) = A;$$

and therefore

$$\lim_{z=z_0}\ \lim_{\substack{x=x_0\\ y=y_0}} f(x,y,z) = A.$$

In a similar manner we can prove that each of the other limits in question equals A, and since all these limits are therefore equal to A, it follows that the theorem is true.

Theorem 4.—If the simultaneous limit in three variables exists and equals A, and if also

$$\left.\begin{array}{lllllll}\lim\limits_{x=x_0} f(x, y, z) & \text{exists for every pair of values} & (y, z) \\ \lim\limits_{y=y_0} f(x, y, z) & \text{`` `` `` `` ``} & (x, z) \\ \lim\limits_{z=z_0} f(x, y, z) & \text{`` `` `` `` ``} & (x, y)\end{array}\right\} \begin{array}{l}\text{in the neighbor-} \\ \text{hood of} \\ (x_0, y_0, z_0),\end{array}$$

then we may also interchange the order of limits and have

$$\lim_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z) = \lim_{z=z_0} \lim_{\substack{x=x_0\\ y=y_0}} f(x, y, z) = \lim_{\substack{x=x_0\\ y=y_0}} \lim_{z=z_0} f(x, y, z) = \text{etc.} = A.$$

The proof of this theorem is similar to that of theorem 3. Since by hypothesis the single limits exist, it follows that $\lim\limits_{z=z_0} f(x, y, z) = F(x, y, z_0)$, for every value of (x, y) where $x_0 < x \leqq x_0 + \delta_1$, and $y_0 < y \leqq y_0 + \delta_2$, and $|\delta_1| \leqq |\delta|$, $|\delta_2| \leqq |\delta|$ simultaneously. Hence

$$|F(x, y, z_0) - f(x, y, z_0 + \delta_3)| < \sigma, \; |\delta_3| \leqq |\delta| \tag{1}$$

Since, furthermore, the simultaneous limit in three variables exists and equals A, we have

$$|f(x, y, z_0 + \delta'_3) - A| < \sigma, \tag{2}$$

for the same σ and every value (x, y) where $x_0 < x \leqq x_0 + \delta'_1$, and $y_0 < y \leqq y_0 + \delta'_2$, and $|\delta'_1| = |\delta|, |\delta'_2| = |\delta|, |\delta'_3| = |\delta|$, simultaneously. Now let $|\delta'_1| = |\delta_1|$, $|\delta'_2| = |\delta_2|, |\delta'_3| = |\delta_3|$; then (1) and (2) will hold for the same δ's, and we may therefore add these inequalities and obtain

$$|F(x, y, z_0) - A| < 2\sigma,$$

for every value (x, y) where $x_0 < x \leqq x_0 + \delta_1$, and $y_0 < y \leqq y_0 + \delta_2$, and where $\delta_1 > 0$ and $\delta_2 > 0$. Hence

$$\lim_{\substack{x=x_0\\ y=y_0}} F(x, y, z_0) = A;$$

and therefore,

$$\lim_{\substack{x=x_0\\ y=y_0}} \lim_{z=z_0} f(x, y, z) = A.$$

In a similar manner we establish the truth of the other limits and our theorem is proved.

3. *Regular Points. Oscillation of a Function at a Point.*—In the preceding sections we have been interested only in the question of the existence of the simultaneous limit in three variables at the point (x_0, y_0, z_0), but we have not said anything about the value of this limit. It was pointed out that the value of the limit depends only upon the value of the function in the neighborhood of the point and not upon the value $f(x_0, y_0, z_0)$ of the function at the point (x_0, y_0, z_0). Now it may happen that the value of the simultaneous limit equals the value of the function at the point, that is $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z) = f(x_0, y_0, z_0)$. In this case we may say that the function $f(x, y, z)$ converges regularly for the value (x_0, y_0, z_0), or, in the language of geometry, that (x_0, y_0, z_0) is a *regular point* of the function. It is evident that at such a point the function is continuous in all three variables taken together, for the condition $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z) = f(x_0, y_0, z_0)$ is exactly the condition that $f(x, y, z)$ is continuous in all three variables taken together. It follows at once from theorem 2 of section 2 that, if $f(x, y, z)$ is continuous in all three variables taken together, then it is also continuous in any two of the variables taken together, and is also continuous in each variable by itself. The converse of these statements is however not true, as is seen from examples 3 and 4 of the preceding section.

If the simultaneous limit in three variables does not exist at the point (x_0, y_0, z_0), then we call the point an *irregular point* of the function. For any continuous approach of (x, y, z) to such a point, one of two things must happen. In the first place we may obtain two different limiting values of the function for two different simultaneous approaches of (x, y, z) to (x_0, y_0, z_0). In this case we say that the function has a *discontinuity of the first kind** at (x_0, y_0, z_0). Examples of such functions have been given in the preceding section. In the second case we may not obtain a definite limiting value of the function at all for any simultaneous approach of (x, y, z) to (x_0, y_0, z_0). In this case we say that the function has a *discontinuity of the second kind*[*] at the point (x_0, y_0, z_0). Thus the function $\sin 1/x + \sin 1/y + \sin 1/z$ has a discontinuity of the second kind at the origin. At such a point the function may be continuous with respect to each variable alone or with respect to any two of the variables taken together, though it will not be continuous with respect to all three taken together.

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*Cf. Dini: Grundlagen für die Theorie der Funktionen einer veränderlichen reellen Grösse; page 51.

Oscillation of a Function.—Let us construct a sphere of radius ρ about the point (x_0, y_0, z_0), and let us denote the upper and lower limits of the values of the function within this sphere by M and m, respectively. We then call the difference $M - m$ the oscillation of the function $f(x, y, z)$ within the sphere with respect to (x, y, z) taken together. The limit of the oscillation, as ρ decreases indefinitely, we define as the amount of the discontinuity of $f(x, y, z)$ in (x, y, z) together at the point (x_0, y_0, z_0). If, therefore, $M - m = k$, then $\lim\limits_{\rho=0} k$ is the amount of the discontinuity of $f(x, y, z)$ in (x, y, z) together at (x_0, y_0, z_0), or simply the (x, y, z) discontinuity of $f(x, y, z)$ at (x_0, y_0, z_0). From the definition follows at once that at a regular point the limit of the oscillation, as ρ decreases indefinitely, is zero and conversely, if at (x_0, y_0, z_0) $\lim\limits_{\rho=0} k = 0$, then the point is a regular point. If, however, $\lim\limits_{\rho=0} k \neq 0$, then the point in question is an irregular point. Now we have just seen that we may have two kinds of irregular points. In future we shall always call $\lim\limits_{\rho=0} k$ the amount of the discontinuity in each of the two cases. The Germans call this "Sprung".

Example 1. Let $f(x, y, z) = \dfrac{xyz}{x^3 + y^3 + z^3}$, where $f(0, 0, 0) = 0$.

This function is continuous in each variable alone at the origin, but the limit of the oscillation in all three variables taken together is $1/3$ at $(0, 0, 0)$; hence the function is discontinuous with respect to (x, y, z) taken together at this point.

Example 2. Let $f(x, y, z) = \dfrac{xyz^2}{x^3 + y^3 + z^6}$, where $f(0, 0, 0) = 0$.

This function is continuous in each variable by itself at $(0, 0, 0)$, but it has at this point an infinite discontinuity in all three variables taken together.

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[*] Cf. Dini: Grundlagen für die Theorie der Funktionen einer veränderlichen reellen Grösse; page 51.

4. *Interchange of the Order of Limits.*—In investigating certain questions, such as the interchange of the order of differentiation and integration, it is very necessary to know under what conditions we may interchange the order of sequence of the double and triple limits to be considered. In the case of a function of two variables it is only necessary to investigate under what conditions $\lim\limits_{x=x_0}\lim\limits_{y=y_0} f(x, y) = \lim\limits_{y=y_0}\lim\limits_{x=x_0} f(x, y)$. In the case of a function of three variables, however, there are six possible cases, but there is considerable similarity among these, so that we need not give an extended proof in every case. For the sake of convenience we shall always take the origin as the critical point, which we may do without loss of generality.

Case 1. The necessary and sufficient condition that

$$\lim_{x=0}\lim_{\substack{y=0\\z=0}} f(x, y, z) = \lim_{\substack{y=0\\z=0}}\lim_{x=0} f(x, y, z) = A$$

is that

(1) $\lim\limits_{\substack{y=0\\z=0}} f(x, y, z) = \phi(x)$ exists for every $x \neq 0$,

$\lim\limits_{x=0} f(x, y, z) = \psi(y, z)$ exists for every $y \neq 0$ and every $z \neq 0$;

(2) $\lim\limits_{\substack{y=0\\z=0}} \psi(y, z) = A$;

(3) that there shall exist a set of pairs of values (y_i, z_k), dense at $(y=0, z=0)$, and corresponding to each pair (y_i, z_k) for which the absolute value $|\sqrt{(y_i^2 + z_k^2)}| < r$, where $r > 0$ is some fixed quantity, there shall exist an interval $(0, \delta_{i,k})$ greater than zero, which may however vary with (y_i, z_k), such that for every value x of this interval we have

$$|f(x, y_i, z_k) - \phi(x)| < \sigma, \text{ where } 0 < x \leqq \delta_{i,k},$$

and σ is an arbitrarily small positive number.

Considered geometrically we wish to determine the conditions under which the limit taken along PEO will equal the limit taken along PAO (see figure 1).

The geometrical interpretation of conditions (1) and (2) is at once apparent from the figure. Condition 3 asserts that corresponding to every point within a circle of radius r about the origin in the YZ-plane there must exist an interval

$\delta_{i,k} = O'R = OK$, such that the value of the function at any point of $O'R$ shall differ from the value of the function at any point of OK by as little as we please.

Proof. We shall first show that the condition stated is a necessary condition, hence our hypothesis is

$$\lim_{x=0}\ \lim_{\substack{y=0\\z=0}} f(x, y, z) = \lim_{\substack{y=0\\z=0}}\ \lim_{x=0} f(x, y, z) = A.$$

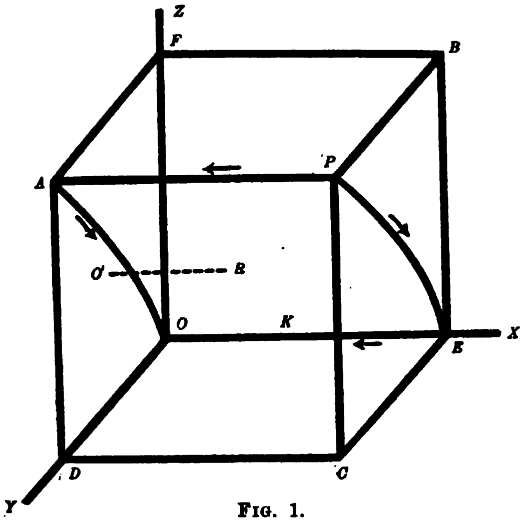

FIG. 1.

If this is so, then we have at once

$$(1) \qquad \lim_{\substack{y=0\\z=0}} f(x, y, z) = \phi(x), \text{ when } x \neq 0,$$

$$\lim_{x=0} f(x, y, z) = \psi(y, z), \text{ when } y \neq 0 \text{ and } z \neq 0;$$

$$(2) \qquad \lim_{\substack{y=0\\z=0}} \psi(y, z) = A, \text{ and } \lim_{x=0} \phi(x) = A;$$

so that the first two conditions of our theorem are fulfilled. Now, since $\lim\limits_{x=0} \phi(x) = A$, we can choose an arbitrarily small $\sigma > 0$ and can then find a δ greater than zero, such that

$$|\phi(x) - A| < 1/3\,\sigma, \text{ where } 0 < x \leqq \delta. \qquad (a)$$

Since we also have $\lim\limits_{\substack{y=0\\z=0}} \psi(y, z) = A$, we can choose the same σ and can then determine a $\theta_1 > 0$ and a $\theta_2 > 0$, such that

$$|A - \psi(y_i, z_k)| < 1/3\,\sigma, \text{ where } 0 < y \leqq \theta_1 \text{ and } 0 < z \leqq \theta_2. \qquad (b)$$

But we have also $\lim\limits_{x=0} f(x, y_i, z_k) = \psi(y_i, z_k)$, hence with the same σ we can determine a $\delta_{i,k} > 0$, such that

$$|\psi(y_i, z_k) - f(x. y_i, z_k)| < 1/3\,\sigma, \text{ where } 0 < x \leqq \delta_{i,k}. \qquad (c)$$

Now let $\delta_{i,k}$ be the smaller of the two intervals δ and $\delta_{i,k}$, and add (a), (b), and (c). We then obtain

$$|\phi(x) - f(x, y_i, z_k)| < \sigma, \text{ where } 0 < x \leqq \delta_{i,k},$$

and this holds for every y_i and z_k different from zero; hence it will hold for a set of pairs of values (y, z), dense at $y = 0$, $z = 0$, which proves our assertion.

We must now prove that the condition is also sufficient. We suppose therefore that the conditions (1), (2), and (3) of the theorem hold. From condition 2 follows at once that, having chosen an arbitrarily small $\sigma > 0$, we can determine a $\theta_1 > 0$ and a $\theta_2 > 0$, such that

$$|\psi(y_i, z_k) - A| < 1/3\,\sigma, \text{ where } 0 < y_i \leqq \theta_1, \text{ and } 0 < z_k \leqq \theta_2. \qquad (a)$$

From condition (3) follows that, having chosen the same σ, we can find a $\delta_{i,k} > 0$, such that

$$|\phi(x) - f(x, y_i, z_k)| < 1/3\,\sigma, \text{ where } 0 < x \leqq \delta_{i,k}, \qquad (b)$$

and where (y_i, z_k) is a pair of values of our set. From condition (1) we finally have that for the same σ, we can find a $\delta > 0$, such that

$$|f(x, y_i, z_k) - \psi(y_i, z_k)| < 1/3\,\sigma, \text{ where } 0 < x \leqq \delta. \qquad (c)$$

Now let $\delta_{i,k}$ be the smaller of the two quantities δ and $\delta_{i,k}$, and add (a), (b), and (c). We thus obtain

$$|\phi(x) - A| < \sigma, \text{ where } 0 < x \leqq \delta_{i,k}.$$

Hence $\lim\limits_{x=0} \phi(x) = A$. But we have by hypothesis that $\phi(x) = \lim\limits_{\substack{y=0\\z=0}} f(x, y, z)$, hence $\lim\limits_{x=0} \lim\limits_{\substack{y=0\\z=0}} f(x, y, z) = A$, which proves the theorem.

In the above theorem no mention was made of the simultaneous limit in three variables; in fact, the theorem enables us to decide when we can interchange the order of sequence of limits even when the simultaneous limit in three variables does not exist.

Example 1. Let $f(x,y,z)=\frac{xyz}{x^3+y^3+z^3}$, where $f(0,0,0)=0$.

Here $\lim\limits_{x=0}\lim\limits_{\substack{y=0\\z=0}}\frac{xyz}{x^3+y^3+z^3}=0$ and $\lim\limits_{y=0}\lim\limits_{\substack{x=0\\z=0}}\frac{xyz}{x^3+y^3+z^3}=0$, though the simultaneous limit $\lim\limits_{\substack{x=0\\y=0\\z=0}} f(x,y,z)$ does not exist.

Case 2. The necessary and sufficient condition that

$$\lim_{x=0}\lim_{\substack{y=0\\z=0}} f(x,y,z)=\lim_{y=0}\lim_{\substack{x=0\\z=0}} f(x,y,z)=A,$$

is that

(1) $\lim\limits_{\substack{y=0\\z=0}} f(x,y,z)=\phi(x)$ for every $x\neq 0$,

$\lim\limits_{\substack{x=0\\z=0}} f(x,y,z)=\psi(y)$ for every $y\neq 0$;

(2) $\lim\limits_{y=0}\psi(y)=A$;

(3) that we have a set of values $y_1, y_2, y_3, \ldots$ dense at $y=0$, and corresponding to each y_n from some y on a region $(0<x\leqq\delta'_n,\ 0<z\leqq\delta''_n)$ of area different from zero, which may however vary with y_n; such that

$$|f(x,y_n,z)-\phi(x)|<\sigma, \text{ where } 0<x\leqq\delta'_n, \text{ and } 0<z\leqq\delta''_n.$$

Considered geometrically, we wish to determine the conditions under which the limit along *PEO* equals the limit along *PDO* (see figure 2). The geometrical interpretation of conditions (1) and (2) is at once evident. Condition (3) asserts that the value of the function at any point of *RSTU* shall differ from the value of the function at any point of *OG* by as little as we please.

Proof. The proof is similar to that of case (1). The condition is necessary; for if

$$\lim_{x=0}\ \lim_{\substack{y=0\\ z=0}} f(x,y,z)=\lim_{y=0}\ \lim_{\substack{x=0\\ z=0}} f(x,y,z)=A,$$

then surely conditions (1) and (2) must hold, and we shall have

$$\lim_{x=0}\ \lim_{\substack{y=0\\ z=0}} f(x,y,z)=\lim_{x=0}\ \phi(x)=A.$$

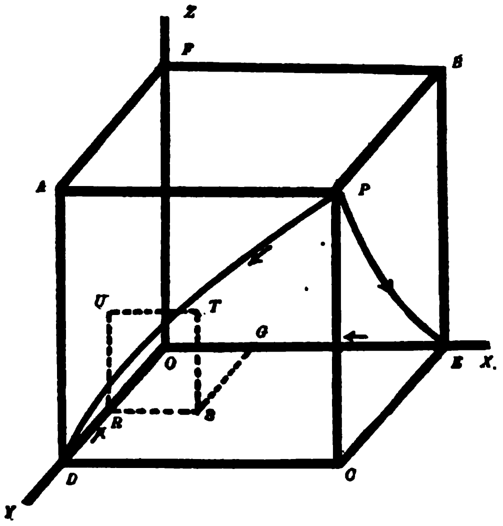

FIG. 2.

Hence we can choose at pleasure a positive number σ and can then find a $\delta>0$, such that

$$|\phi(x)-A|<1/3\,\sigma,\ \text{where}\ 0<x\leqq\delta. \qquad (a)$$

Since $\lim\limits_{y=0}\psi(y)=A$, we can take the same σ and can then find a $\theta>0$, such that

$$|A-\psi(y_n)|<1/3\,\sigma,\ \text{where}\ 0<y_n\leqq\theta, \qquad (b)$$

where y_n belongs to our set. Finally since $\lim\limits_{\substack{x=0\\ z=0}} f(x,y_n,z)=\psi(y_n)$, we can take the same σ and can then find a $\delta'_n>0$ and a $\delta''_n>0$, such that

$$|\psi(y_n)-f(x,y_n,z)|<1/3\,\sigma,\ \text{where}\ 0<x\leqq\delta'_n,\ \text{and}\ 0<z\leqq\delta''_n. \qquad (c)$$

Let δ'_n be the smaller of the two numbers δ and δ'_n. We can then add (a), (b), and (c), and obtain

$$|\phi(x)-f(x,y_n,z)|<\sigma, \text{ where } 0<x\leqq\delta'_n, \text{ and } 0<z\leqq\delta''_n,$$

for every x and z within the region.

The condition is also sufficient; for we have from (2) that, having chosen an arbitrarily small $\sigma>0$, we can then find a $\theta>0$, such that

$$|\psi(y_n)-A|<1/3\sigma, \text{ where } 0<y_n\leqq\theta, \tag{a}$$

where y_n belongs to our set. From (1) we have that with the same σ we can determine a δ' and a δ'', each greater than zero, such that

$$|f(x,y_n,z)-\psi(y_n)|<1/3\sigma, \tag{b}$$

where $0<x\leqq\delta'$ and $0<z\leqq\delta''$, and y_n is some number of our set. Finally we have from (3) that with the same σ we can find a $\delta'_n>0$ and a $\delta''_n>0$, such that

$$|\phi(x)-f(x,y_n,z)|<1/3\sigma, \tag{c}$$

where $0<x\leqq\delta'_n$ and $0<z\leqq\delta''_n$. Suppose now that $\delta'_n<\delta'$ and $\delta''_n<\delta''$. We can then add (a), (b), and (c), and thus obtain

$$|\phi(x)-A|<\sigma, \text{ where } 0<x\leqq\delta'_n,$$

and hence it follows that $\lim\limits_{x=0}\phi(x)=A$. From this we have at once that $\lim\limits_{x=0}\lim\limits_{\substack{y=0\\z=0}}f(x,y,z)=A$, which proves the theorem.

Case 3. The necessary and sufficient condition that

$$\lim_{\substack{y=0\\z=0}}\lim_{x=0}f(x,y,z)=\lim_{\substack{x=0\\z=0}}\lim_{y=0}f(x,y,z)=A,$$

is that

(1) $\lim\limits_{y=0}f(x,y,z)=\phi(x,z)$, where $x\neq0$ and $z\neq0$.

$\lim\limits_{x=0}f(x,y,z)=\psi(y,z)$, where $y\neq0$ and $z\neq0$;

(2) $\lim\limits_{\substack{y=0\\x=0}}\psi(y,z)=A$;

(3) that we have a set of pairs of values (y_i,z_k) dense at $y=0$, $z=0$, and corresponding to each pair for which the absolute value $|\sqrt{(y_i^2+z_k^2)}|<r$, where

r is some fixed positive number, an interval $\delta_{i,k}$ different from zero, which may however vary with (y_i, z_k), such that

$$|f(x, y_i, z_k) - \psi(x, z)| < \sigma,$$

where $0 < x \leqq \delta_{i,k}$, and $0 < z \leqq z_k$.

Considered geometrically we wish to determine the conditions under which the limit along PEO will equal the limit taken along PAO (see figure 3). The geometrical interpretation of conditions (1) and (2) is at once apparent from the figure. Condition (3) asserts that the value of the function at any point of $\delta_{i,k} = O'G$ shall differ by as little as we please from the value of the function at any point of $OHIK$, boundaries excluded. The proof in this case is exactly analogous to that of cases 1 and 2 and is therefore omitted.

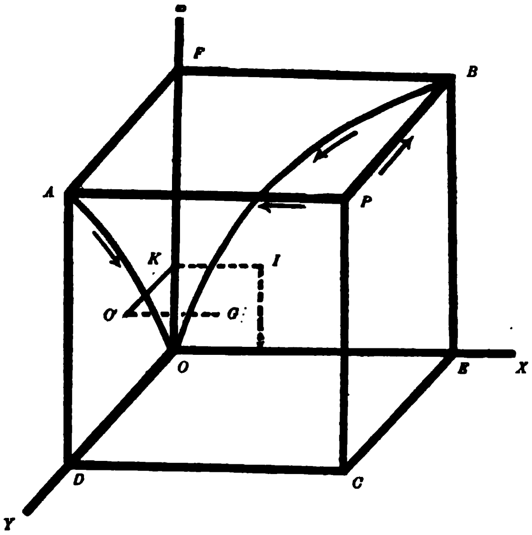

FIG. 3.

Case 4. The necessary and sufficient condition that

$$\lim_{y=0} \lim_{\substack{x=0\\z=0}} f(x, y, z) = \lim_{\substack{y=0\\z=0}} \lim_{x=0} f(x. y, z) = A,$$

is that

(1) $$\lim_{\substack{x=0\\z=0}} f(x, y, z) = \phi(y), \text{ where } y \neq 0,$$

$$\lim_{x=0} f(x, y, z) = \psi(y, z), \text{ where } y \neq 0 \text{ and } z \neq 0;$$

(2) $$\lim_{\substack{y=0\\z=0}} \psi(y, z) = A;$$

(3) that there exists a set of pairs of values (y_i, z_k) dense at $y=0$, $z=0$, and corresponding to each pair of values for which $|\sqrt{(y_i^2+z_k^2)}| < r$, where r is some fixed positive number, an interval $(0, \delta_{i,k})$ different from zero, which may however vary with (y_i, z_k), such that

$$|\phi(y) - f(x, y_i, z_k)| < \sigma, \text{ where } 0 < x \leqq \delta_{i,k}.$$

Geometrically we are to determine the conditions under which the limit taken along *PDO* shall equal the limit taken along *PAO* (see figure 4). In this case the proof is exactly analogous to that of the three preceding cases and is therefore omitted.

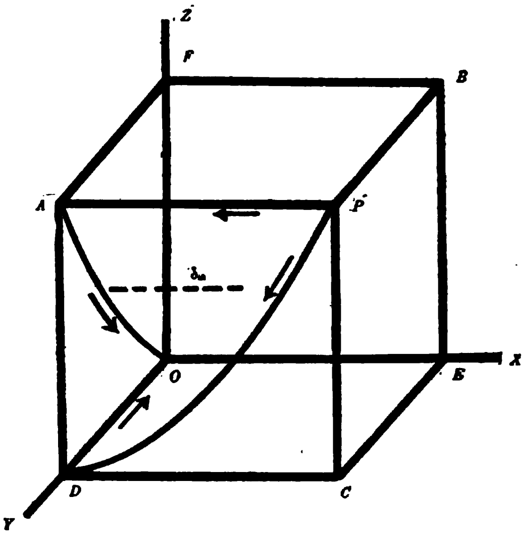

FIG. 4.

These four cases are those that involve both simple and simultaneous limits in two variables. We have still to consider the cases which involve only simple limits. Here we have two possibilities.

Case 5. The necessary and sufficient condition that

$$\lim_{x=0} \lim_{y=0} \lim_{z=0} f(x, y, z) = \lim_{z=0} \lim_{y=0} \lim_{x=0} f(x, y, z) = A,$$

is that

(1) $\lim\limits_{z=0} f(x, y, z) = \phi_1(x, y)$, where $x \neq 0$ and $y \neq 0$,

$\lim\limits_{y=0} \phi_1(x, y) = \psi_1(x)$, where $x \neq 0$,

$\lim\limits_{x=0} f(x, y, z) = \phi_2(y, z)$, where $y \neq 0$ and $z \neq 0$,

$\lim\limits_{y=0} \phi_2(y, z) = \psi_2(z)$, where $z \neq 0$;

(2) $\lim\limits_{z=0} \psi_2(z) = A$;

(3) that there shall exist a set of values $z_1, z_2, z_3 \ldots.$, dense at $z = 0$, and corresponding to each z_n from some z on an interval $(0, \delta_n)$ different from zero, but which may vary with z_n and that there shall also exist a region $C(0 < y \leqq y_i, 0 < z \leqq z_k)$, such that

$$|\psi_1(x) - f(x, y, z)| < \sigma, \text{ where } 0 < x \leqq \delta_n \text{ and } (y, z) \text{ is a point of the region } C.$$

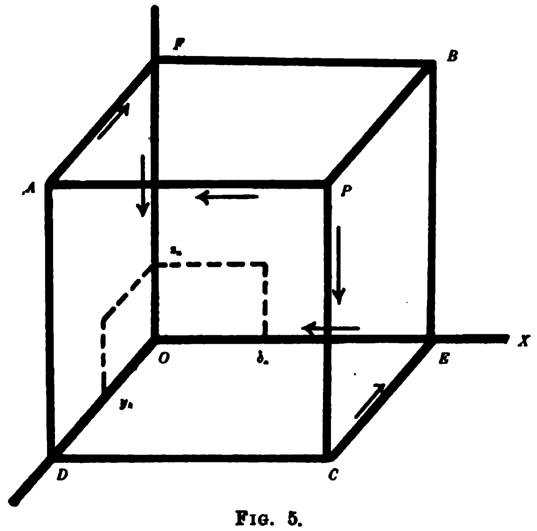

FIG. 5.

Geometrically we are to determine the conditions under which the limit taken along *PCEO* equals the limit taken along *PAFO* (see figure 5).

Proof. The condition is necessary; for if

$$\lim_{x=0} \lim_{y=0} \lim_{z=0} f(x, y, z) = \lim_{z=0} \lim_{y=0} \lim_{x=0} f(x, y, z) = A,$$

then we surely have

$$\lim_{x=0}\ \lim_{y=0}\ \lim_{z=0} f(x, y, z) = \lim_{x=0}\ \lim_{y=0} \phi_1(x, y) = \lim_{x=0} \psi_1(z) = A,$$

and also

$$\lim_{z=0}\ \lim_{y=0}\ \lim_{x=0} f(x, y, z) = \lim_{z=0}\ \lim_{y=0} \phi_2(y, z) = \lim_{z=0} \psi_2(z) = A.$$

From these equalities we have at once that, having chosen an arbitrarily small $\sigma > 0$, we can then find a $\delta > 0$, such that

$$|\psi_1(x) - A| < 1/4\sigma, \text{ where } 0 < x \leqq \delta. \qquad (a)$$

With the same σ we can also find a $z'_n > 0$, such that

$$|A - \psi_2(z)| < 1/4\sigma, \text{ where } 0 < z \leqq z'_n. \qquad (b)$$

Again with the same σ we can determine a $y'_k > 0$ and a $z''_n > 0$, such that

$$|\psi_2(z) - \phi_2(y, z)| < 1/4\sigma, \text{ where } 0 < y \leqq y'_k \text{ and } 0 < z \leqq z''_n. \qquad (c)$$

Finally we can find a $\delta_n > 0$ and a $y_i > 0$ and a $z_k > 0$, such that

$$|\phi_2(y, z) - f(x, y, z)| < 1/4\sigma, \qquad (d)$$

where $0 < x \leqq \delta_n$, $0 < y \leqq y_i$, and $0 < z \leqq z_k$. Now let $\delta_n \leqq \delta$, $y_i \leqq y'_k$, and z_k be less than either z'_n or z''_n. Then we may add (*a*), (*b*), (*c*), and (*d*), and thus obtain

$$|\psi_1(x) - f(x, y, z)| < \sigma.$$

This proves that the condition is necessary. But the condition is also sufficient; for it follows from (2) that, having chosen at pleasure a $\sigma > 0$, we can then find a $z_k > 0$, such that

$$|\psi_2(z_n) - A| < 1/4\sigma, \text{ where } 0 < z_n \leqq z_k. \qquad (a)$$

From (1) follows that for the same σ we can find a $y_i > 0$ and also a $\delta > 0$, such that

$$|\phi_2(y_k, z_n) - \psi_2(z_n)| < 1/4\sigma. \text{ where } 0 < y_k \leqq y_i, \qquad (b)$$

and

$$|f(x, y_k, z_n) - \phi_2(y_k, z_n)| < 1/4\sigma, \text{ where } 0 < x \leqq \delta. \qquad (c)$$

Finally we have from (3) that for the same σ we can find a $\delta_n > 0$, such that

$$|\psi_1(x) - f(x, y_k, z_n)| < 1/4\,\sigma, \text{ where } 0 < x \leqq \delta_n. \qquad (d)$$

Now let δ_n be the smaller of δ_n and δ. Then we obtain by the addition of (*a*), (*b*), (*c*), and (*d*)

$$|\psi_1(x) - A| < \sigma, \text{ where } 0 < x \leqq \delta_n,\ 0 < y \leqq y_i,\ 0 < z \leqq z_k.$$

Hence $\lim\limits_{x=0} \psi_1(x) = A$, and therefore

$$\lim_{x=0}\ \lim_{y=0}\ \phi_1(x, y) = \lim_{x=0}\ \lim_{y=0}\ \lim_{z=0}\ f(x, y, z) = A,$$

which proves the theorem.

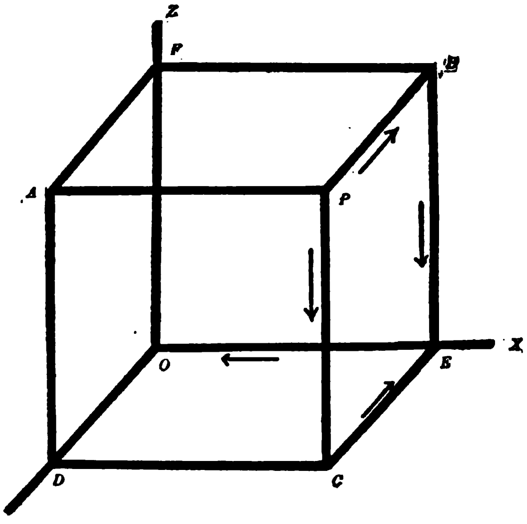

FIG. 6.

Case 6. This is the last possible case where we have to determine the conditions under which the limit taken along *PBEO* equals the limit taken along *PCEO* (see figure 6).

We are to find the conditions under which

$$\lim_{x=0}\ \lim_{y=0}\ \lim_{z=0}\ f(x, y, z) = \lim_{x=0}\ \lim_{z=0}\ \lim_{y=0}\ f(x, y, z) = A.$$

It is at once evident that this condition is the same as is the condition that

$$\lim_{y=0}\ \lim_{z=0}\ f(x_k, y, z) = \lim_{z=0}\ \lim_{y=0}\ f(x_k, y, z) \text{ for any } x_k.$$

It is therefore a question of simultaneous limits in two variables which we state.

The necessary and sufficient condition that

$$\lim_{y=0}\ \lim_{z=0} f(x, y, z) = \lim_{z=0}\ \lim_{y=0} f(x, y, z) = F(x),$$

for every x different from zero, is that for every such x we have

(1) $\lim\limits_{z=0} f(x, y, z) = \phi(x, y)$, where $y \neq 0$,

$\lim\limits_{y=0} f(x, y, z) = \psi(x, z)$, where $z \neq 0$;

(2) $\lim\limits_{z=0} \psi(x, z) = F(x)$;

(3) that for every such x there shall exist a set of numbers $y_1, y_2, y_3 \cdots$, dense at $y = 0$, and corresponding to each y of the set an interval $(0, \delta_i)$, such that

$$|\phi(x, y) - F(x)| < \sigma, \text{ where } 0 < z \leqq \delta_i.$$

CHAPTER II.

PROPERTIES OF FUNCTIONS OF THREE VARIABLES.

5. *Continuity.*—In the case of a function of three variables we may have three different kinds of continuity.

(1) The function $f(x, y, z)$ may be continuous in each variable alone. That is to say, if we consider any two of the variables as constant, the resulting function of the third variable may be continuous in that variable alone.

(2) The function $f(x, y, z)$ may be continuous in two of the variables taken together. For example, if z is supposed to be constant, the resulting function of x and y may be continuous in (x, y) taken together. We say then that the function is continuous in two variables taken together, if the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0}} f(x, y, z)$ exists and equals the value of the function at the point in question.

(3) The function $f(x, y, z)$ may be continuous in all three variables taken together. We define this as follows: If about the point (x_0, y_0, z_0) we construct a sphere of radius ρ and denote the oscillation of the function within this sphere by k, and if we then have

$$\lim_{\rho=0} k = 0,$$

then we say that $f(x, y, z)$ is continuous in all three variables taken together at the point (x_0, y_0, z_0). As we have already seen in section 3, the point (x_0, y_0, z_0) is in this case a regular point of the function. We may therefore also say that $f(x, y, z)$ is continuous in all three variables taken together at (x_0, y_0, z_0), if the simultaneous limit in three variables $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals $f(x_0, y_0, z_0)$.

Points at which the limit of the oscillation, as ρ approaches zero, is different from zero have already been defined as irregular points. We may also call them points of discontinuity. The value of the limit of the oscillation k as ρ approaches zero we shall call the amount of the discontinuity. It is at once evident that if the function is discontinuous in any variable alone or in any two variables taken together, then it cannot be continuous in all three variables taken together. It is also evident that if the function is continuous in each variable alone, or if it is continuous in any two variables taken together, then it does not follow that it will be continuous in all three variables taken together. This is shown in some of the following examples.

Example 1. Let $f(x, y, z) = \dfrac{xyz}{x^3 + y^3 + z^3}$, where $f(0, 0, 0) = 0$.

At the origin this function is continuous in each variable alone and also in any two of the variables taken together, but it is not continuous in all three variables taken together, for the simultaneous limit $\lim\limits_{\substack{x=0\\ y=0\\ z=0}} f(x, y, z)$ does not exist.

Example 2. Let $f(x, y, z) = \dfrac{xy}{z}$, where $f(0, 0, 0) = 0$.

At the origin this function is not continuous in (x, y) taken together, but it is continuous in z alone, since when $x = 0$ and $y = 0$ we have $\lim\limits_{z=0} xy/z = 0$. But when $x \neq 0$ and $y = 0$, then $\lim\limits_{z=0} xy/z$ becomes infinite. Hence the function is not continuous at the origin in all three variables taken together.

Example 3. Let $f(x, y, z) = \dfrac{xy + z}{xy - z}$, where $f(0, 0, 0) = 1$.

At the origin this function is continuous in (x, y) together, for when $z = 0$, $\lim\limits_{\substack{x=0\\y=0}} f(x, y, z) = 1 = f(0, 0, 0)$. When, however, $z \neq 0$, then $\lim\limits_{\substack{x=0\\y=0}} f(x, y, z) = -1$. Hence the function is not continuous in z alone and it cannot therefore be continuous at the origin in all three variables taken together.

Example 4. Let $f(x, y, z) = \frac{xyz}{xy + z^2}$, where $f(0, 0, 0) = 0$.

This function is continuous at the origin in all three variables taken together, for the simultaneous limit $\lim\limits_{\substack{x=0\\y=0\\z=0}} f(x, y, z) = 0$.

In all of the above examples we have dealt only with the continuity of a function at a point. In the following section we shall study the question of the continuity of a function in all three variables taken together throughout an entire region.

6. *Continuity of a Function Throughout a Region.*—In this and the next section we take up theorems that have to do with functions which are continuous throughout a region. We shall always assume that the function is singly determinate throughout the region, but for the present at least we shall not consider the behavior of the function on the boundaries of the region. We, therefore, define our region as follows:

$$\alpha < x < \beta, \quad a < y < b, \quad z_0 < z < z_n.$$

It is not always necessary to consider all values of z within the above region, but it is sufficient for many purposes to take into consideration only a set of z's which is enumerable and dense at $z = z_0$. Unless, however, something is said to the contrary, we shall assume that z takes on all values of the above interval. With this understanding we now proceed to prove the following theorems.

Theorem 1.—If the function $f(x, y, z)$ is continuous in two variables taken together and in the third variable alone throughout the region above defined, and if (x_0, y_0, z_0) is any point of the region, then we have

$$\lim_{\substack{x=x_0\\y=y_0}} \lim_{z=z_0} f(x, y, z) = \lim_{z=z_0} \lim_{\substack{x=x_0\\y=y_0}} f(x, y, z) = \lim_{\substack{x=x_0\\z=z_0}} \lim_{y=y_0} f(x, y, z) =$$

$$\lim_{y=y_0} \lim_{\substack{x=x_0\\z=z_0}} f(x, y, z) = \lim_{\substack{y=y_0\\z=z_0}} \lim_{x=x_0} f(x, y, z) = \lim_{x=x_0} \lim_{\substack{y=y_0\\z=z_0}} f(x, y, z) = f(x_0, y_0, z_0).$$

Since $f(x, y, z)$ is continuous in (x, y) taken together, we can consider z as constant and shall then have

$$\lim_{\substack{x=x_0\\ y=y_0}} f(x, y, z) = f(x_0, y_0, z),$$

and this will hold for every z of the interval $z_0 < z < z_n$. But we have by hypothesis that $f(x_0, y_0, z)$ is continuous in z alone, when we consider z as variable, and hence

$$\lim_{z=z_0} f(x_0, y_0, z) = f(x_0, y_0, z_0).$$

This gives us

$$\lim_{z=z_0} \lim_{\substack{x=x_0\\ y=y_0}} f(x, y, z) = \lim_{z=z_0} f(x_0, y_0, z) = f(x_0, y_0, z_0),$$

which proves our theorem for this case. The other cases are proved in a similar manner.

Theorem 2.—If the function $f(x, y, z)$ is continuous in each variable alone throughout the region above defined and if (x_0, y_0, z_0) is any point of the region, then we have $\lim\limits_{x=x_0} \lim\limits_{y=y_0} \lim\limits_{z=z_0} f(x, y, z) = \lim\limits_{x=x_0} \lim\limits_{z=z_0} \lim\limits_{y=y_0} f(x, y, z) =$ $\lim\limits_{y=y_0} \lim\limits_{x=x_0} \lim\limits_{z=z_0} f(x, y, z) = \lim\limits_{y=y_0} \lim\limits_{z=z_0} \lim\limits_{x=x_0} f(x, y, z) = \lim\limits_{z=z_0} \lim\limits_{x=x_0} \lim\limits_{y=y_0} f(x, y, z) =$ $\lim\limits_{z=z_0} \lim\limits_{y=y_0} \lim\limits_{x=x_0} f(x, y, z) = f(x_0, y_0, z_0)$.

The proof is similar to that of theorem 1. Let us first consider x and y as constants; then, since the resulting function of z is continuous in z, we have

$$\lim_{z=z_0} f(x, y, z) = f(x, y, z_0),$$

and this will hold for every x and y of the respective intervals. But we have also by hypothesis that $f(x, y, z_0)$ is continuous in y alone, and hence if we consider x as constant, we shall have

$$\lim_{y=y_0} f(x, y, z_0) = f(x, y_0, z_0),$$

and this will hold for every x of the interval $\alpha < x < \beta$. Finally, since $f(x, y_0, z_0)$ is continuous in x alone, we have

$$\lim_{x=x_0} f(x, y_0, z_0) = f(x_0, y_0, z_0),$$

which proves our theorem for this case. The other cases are proved in a like manner.

Theorem 3.—If the function $f(x, y, z)$ is, throughout the region $\alpha < x < \beta$, $a < y < b$, $z_0 < z < z_n$, continuous in x alone, y alone, and z alone, and if we have a set of values (x, y, z) everywhere dense, in at least a part of the given region, in such a manner that they shall be dense at (x_0, y_0, z_0) along the lines $x = x_0, y = y_0, z = z_0$, where (x_0, y_0, z_0) is one of the limiting points, and if we let (x, y, z) approach (x_0, y_0, z_0) through this set of values in such a manner that no two of the variables vary together, then we always obtain the limiting value $f(x_0, y_0, z_0)$. Moreover, if we choose a second set of values (x, y, z) which shall be everywhere dense in the same subregion, and dense at (x_0, y_0, z_0) along the lines $x = x_0$, $y = y_0$, $z = z_0$, and let (x, y, z) approach (x_0, y_0, z_0) through this second set of values, then we obtain the same limiting value $f(x_0, y_0, z_0)$ as before.

The proof of this theorem follows from the fact that the limiting value of the function $f(x, y, z)$ at the point (x_0, y_0, z_0) depends only upon those values of the function in the neighborhood of (x_0, y_0, z_0) through which the function approaches its limiting value. Since now by hypothesis no two of the variables x, y, and z, are to vary together, the function $f(x, y, z)$ must eventually approach its limiting value in such way that two of the variables will be constant, for example $x = x_0$ and $y = y_0$. Then $f(x_0, y_0, z)$ is a continuous function of z, and hence we obtain the same limiting value $f(x_0, y_0, z_0)$, no matter through what set of points z approaches z_0. This proves our assertion. In a like manner we can prove it for the other cases.

7. *Continuity of a Function.*—(*Continued.*)—In this section we shall consider the number and the nature of the points within a given region, at which a function of three variables is continuous or discontinuous in all three variables taken together. As before, we shall not consider the behavior of the function on the boundaries of the region.

Theorem 1.[*]—If the points, at which the discontinuity of $f(x, y, z)$ in (x, y, z) taken together is greater than or equal to a given positive number σ, become dense at (x_0, y_0, z_0), then $f(x, y, z)$ is discontinuous in (x, y, z) taken together, also at (x_0, y_0, z_0), and the (x, y, z) discontinuity at (x_0, y_0, z_0) is greater than or equal to σ.

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[*] Cf. Schoenfliess: Goettinger Nachrichten, 1899, page 187.

In order to prove this theorem we need the following facts. If about the point P as center we have a sphere of radius ρ, then, as before, we denote the oscillation of the function within this sphere by k and the limit of the oscillation as ρ approaches zero by k'. If now $\bar{k}$ is some positive number greater than k', then it is evident that we can find about P some sphere within which the oscillation k of the function is less than $\bar{k}'$. On the other hand, if $\bar{k}'$ is some positive number less than k', then we cannot find any sphere about P, such that within this sphere the oscillation k of the function is less than $\bar{k}'$.

This being established, we can prove our theorem without any difficulty. Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, ... be a set of points dense at $P_0(x_0, y_0, z_0)$. Suppose now that the theorem is not true. If now we construct about P_0 as center a sphere of radius ρ, then the limit k' of the oscillation of the function as ρ approaches zero will be less than σ. We can now find a positive number σ' satisfying the relation $k' < \sigma' < \sigma$. From the facts given at the beginning of the proof it therefore follows that we can find some sphere of radius r about P_0, such that within this sphere the oscillation k of the function is everywhere less than σ'. This sphere will surely contain some points of the set P_1, P_2, P_3, ... since these points are dense at P_0. Let P_k be one of the points within this sphere. Then at P_k the limit k' of the oscillation of the function is less than σ', that is, it is also less than σ, which is contrary to the hypothesis. Hence we are led to a contradiction, and the theorem must be true.

We next define what we mean when we say that a function is *pointwise discontinuous*. In this we follow Schoenfliess.*

If a function $f(x, y, z)$ is discontinuous in (x, y, z) taken together at a finite or an infinite set of points, but if at the same time it is continuous in (x, y, z) taken together at a set of points that is everywhere dense in the region in question, then we say that the function is pointwise discontinuous in (x, y, z) taken together.

Theorem 2.*—The necessary and sufficient condition that a function $f(x, y, z)$ is pointwise discontinuous in (x, y, z) taken together is that the points at which the amount of the (x, y, z) discontinuity is greater than or equal to σ are not everywhere dense in any subregion.

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* Cf. Schoenfliess: Jahresbericht der Deutschen Mathematiker-Vereinigung. Band 8, (1900), page 128.

The condition is necessary; for if the points at which the limit k' of the oscillation is greater than or equal to σ are everywhere dense in any subregion, then the limit k' of the oscillation would be greater than or equal to σ also at the limiting points, hence this limit k' would be greater than or equal to σ at every point of the subregion, which is contrary to the hypothesis.

The condition is also sufficient; for let us choose a set of positive quantities

$$\sigma_0 > \sigma_1 > \sigma_2 > \dots\dots > \sigma_n > \dots\dots$$

where

$$\lim_{n=\infty} \sigma_n = 0.$$

Corresponding to each of these quantities σ_i choose a set of points P_i which shall not be everywhere dense in any subregion, and such that at each of the points of P_i the amount k' of the (x, y, z) discontinuity is greater than or equal to σ_i, where $i = 1, 2, 3, \dots, i, \dots$ We thus obtain a sequence of sets of points

$$P_0, P_1, P_2, P_3, \dots\dots, P_i \dots\dots$$

Each of these sets certainly contains all of the points of all of the preceding sets. Now, since by hypothesis the points P_i at which the amount k' of the (x, y, z) discontinuity is greater than or equal to σ_i are not everywhere dense in any subregion, it follows that the points at which $k' < \sigma_i$ must be everywhere dense in every subregion, that is to say, in the whole region. We wish to show that the points at which $k' = 0$, as n increases indefinitely, are dense everywhere in the region. In order to do this, let R_0 be a sphere lying entirely within our region. Since the points of P_0 are not everywhere dense in any subregion, it follows either that the sphere R_0 contains no points of P_0 or that there exists a sphere R_1 lying entirely within R_0 such that it contains no points of P_0. Again the sphere R_1 either contains no points of P_1 or there exists a sphere R_2 lying wholly within R_1 which contains no points of P_1. Repeat this reasoning. We thus obtain a set of spheres $R_0, R_1, R_2, \dots, R_n$, each lying wholly within all of the preceding spheres, and such that the sphere R_n contains no points of the set P_{n-1}. If now we let n increase indefinitely, we obtain a limiting point P at which $k' = 0$, and hence at this point the function $f(x, y, z)$ is continuous in (x, y, z) taken together. It follows from this that within our region we can find no subregion which is free from points at which $f(x, y, z)$ is continuous in all three variables taken together. Hence our function can be at most pointwise discontinuous in all three variables together.

Theorem 3.—If throughout the region $\alpha < x < \beta$, $a < y < b$, $z_0 < z < z_n$, the function $f(x, y, z)$ is continuous in two variables taken together and in the third variable by itself, then $f(x, y, z)$ can be at most pointwise discontinuous in all three variables taken together, that is to say, the points at which the (x, y, z) discontinuity is greater than or equal to some positive number $\bar{k}$ cannot be everywhere dense in any subregion.

Suppose that within our region there exists a subregion

$$\alpha' < x < \beta', \ a' < y < b', \ z'_0 < z < z'_n,$$

within which the (x, y, z) discontinuity is greater than or equal to $\bar{k}$ at a set of points everywhere dense. Then it follows from theorem 1 that the function $f(x, y, z)$ must have an (x, y, z) discontinuity, greater than or equal to $\bar{k}$ at every point of this subregion. Let $P_0(x_0, y_0, z_1)$ be some point of this subregion and consider the plane $z = z_1$ parallel to the XY-plane (see figure 7). Since $f(x, y, z)$ is continuous in (x, y) together throughout the entire region, we can choose at pleasure a positive number σ and can then construct in the plane $z = z_1$ about P_0 a circle C_1 of radius r_1, such that for every point of this circle we have

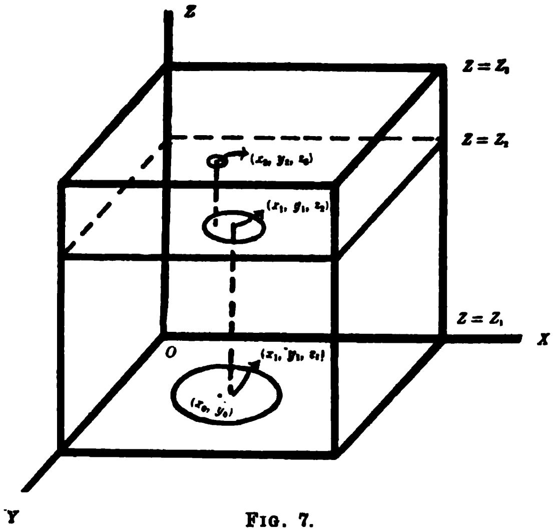

FIG. 7.

$$|f(x, y, z_1) - f(x_0, y_0, z_1)| < \sigma. \tag{1}$$

By hypothesis we have that at every point of the subregion the (x, y, z) discontinuity is greater than or equal to $\bar{k}$ so that the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_1}} f(x, y, z)$ does not exist. We therefore can construct about P_0 as center a sphere R_1 of radius ρ_1 which we may choose less than r_1, such that for some points of this sphere we have

$$|f(x, y, z) - f(x_0, y_0, z_1)| > 1/3\,\bar{k}.$$

Let (x_1, y_1, z_2) be such a point of the sphere R_1, so that

$$|f(x_0, y_0, z_1) - f(x_1, y_1, z_2)| > 1/3\,\bar{k}. \tag{2}$$

By hypothesis $f(x, y, z)$ is continuous in (x, y) together, also at every point of the plane $z=z_2$, as far as it lies within the subregion. We can therefore construct in this plane about (x_1, y_1) a circle C_2 of radius $r_2<r_1$, lying wholly within the circle C_1 of radius r_1, such that for every point (x, y) of this circle we have

$$|f(x_1, y_1, z_2) - f(x, y, z_2)| < \sigma. \tag{3}$$

If we choose σ small enough and combine (1), (2), and (3), we obtain

$$|f(x, y, z_1) - f(x, y, z_2)| \geqq 1/4\,\bar{k}, \tag{4}$$

and this will hold for every point of the circle C_2 of radius r_2 in the plane $z=z_2$. Now let $(\bar{x}_2, \bar{y}_2)$ be some point within the circle C_2 of radius r_2, in the plane $z=z_2$. Since the (x, y, z) discontinuity of the function $f(x, y, z)$ at the point $(\bar{x}_2, \bar{y}_2, z_1)$ is greater than or equal to $\bar{k}$, we can construct about $(\bar{x}_2, \bar{y}_2, z_1)$ a sphere R_2 of radius ρ_2 which we choose less than r_2, such that for some points of this circle we have

$$|f(x, y, z) - f(\bar{x}_2, \bar{y}_2, z_1)| > 1/3\,\bar{k}.$$

Let (x_2, y_2, z_3) be such a point of the sphere R_2, so that

$$|f(x_2, y_2, z_3) - f(\bar{x}_2, \bar{y}_2, z_1)| > 1/3\,\bar{k}. \tag{5}$$

But the function $f(x, y, z)$ is continuous in (x, y) together at every point of the plane $z=z_3$, as far as it lies within our region; hence we can construct a circle C_3 of radius $r_3<r_2$ about (x_2, y_2) in the plane $z=z_3$, which circle shall lie wholly within the circle C_2 of radius r_2, such that for every point of this circle

$$|f(x, y, z_3) - f(x_2, y_2, z_3)| < \sigma. \tag{6}$$

From the continuity of $f(x, y, z)$ in (x, y) together we also have

$$|f(\bar{x}_2, \bar{y}_2, z_1) - f(x, y, z_1) < \sigma, \tag{7}$$

for all points of the circle C_3 of radius r_3 in the plane $z=z_2$. If now we choose σ small enough, we obtain by combining (5), (6), and (7),

$$|f(x, y, z_1)-f(x, y, z_3)| \geqq 1/4\,\bar{k}. \tag{8}$$

Continuing this reasoning, we obtain in general

$$|f(x, y, z_1)-f(x, y, z_n)| \geqq 1/4\,\bar{k}, \tag{9}$$

where (x, y) is any point of the circle C_n of the radius r_n, this circle lying wholly within each of a set of circles of radii

$$r_1 > r_2 > r_3 \ldots\ldots > r_n > \ldots.$$

The centers of these circles therefore approach a definite limiting point, say $(\bar{x}_0, \bar{y}_0)$. This point lies within each of the circles. Now we have by hypothesis that $\lim\limits_{\substack{x=\bar{x}_0\\ y=\bar{y}_0}} f(x, y, z_0) = f(\bar{x}_0, \bar{y}_0, z_0)$. But from (9) we have

$$|f(\bar{x}_0, \bar{y}_0, z_n)-f(\bar{x}_0, \bar{y}_0, z_0)| \geqq 1/4\,\bar{k}, \tag{10}$$

where $z_1, z_2, z_3, \ldots\ldots, z_n, \ldots.$ have the limiting value z_0. Hence we have from (10)

$$\lim_{z=z_0} f(x, y, z) \neq f(x, y, z_0)$$

for $x=\bar{x}_0$ and $y=\bar{y}_0$, which is contrary to the hypothesis that

$$\lim_{z=z_0} f(x, y, z) = f(x, y, z_0)$$

for every constant x and y. The assumption we made is therefore false and the theorem is true.

In the preceding theorem we have considered only the (x, y, z) discontinuities greater than or equal to some positive number $\bar{k}$. Let us now remove this restriction and consider all (x, y, z) discontinuities. We then have the following theorem.

Theorem 4.—If the function $f(x, y, z)$ is continuous in two variables taken together and in the third variable alone, then it may be discontinuous in all three variables taken together at a set of points that is everywhere dense, but the amount of the discontinuity cannot be greater than or equal to $\bar{k}$ at each of these points.

We shall prove the first part of this theorem by means of an example.

Example. Let $\phi(x, y, z) = \dfrac{xyz}{x^3+y^3+z^3}$, where $\phi(0, 0, 0) = 0$ and

$$-1 < x \leqq 1, \quad -1 < y \leqq 1, \quad -1 < z \leqq 1.$$

It has already been shown that this function is continuous in each variable alone and also in any two variables taken together, but it is not continuous in all three variables taken together at the point $(0, 0, 0)$, where the amount of the (x, y, z) discontinuity is $1/3$. Now let us form

$$\phi_k(x, y, z) = \frac{\sin^2 k!\,\pi x \sin^2 k!\,\pi y \sin^2 k!\,\pi z}{\sin^6 k!\,\pi x + \sin^6 k!\,\pi y + \sin^6 k!\,\pi z}.$$

This function is also continuous in each variable alone and in any two of the variables taken together, but it is discontinuous in all three variables taken together. Now let $k = 1$, so that

$$\phi_1(x, y, z) = \frac{\sin^2 \pi x \sin^2 \pi y \sin^2 \pi z}{\sin^6 \pi x + \sin^6 \pi y + \sin^6 \pi z}.$$

This function is discontinuous in (x, y, z) taken together at the points for which x equal 0 or 1, y equal 0 or 1, z equal 0 or 1, that is to say at the points $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$, $(0, 1, 1)$, $(1, 0, 0)$, $(1, 0, 1)$, $(1, 1, 0)$, $(1, 1, 1)$; for at these points the simultaneous limit $\lim\limits_{\substack{x = x_0 \\ y = y_0 \\ z = z_0}} \phi_1(x, y, z)$ does not exist. At all other points $\phi_1(x, y, z)$ is continuous in all three variables taken together. In a like manner we see that $\phi_k(x, y, z)$ is discontinuous in all three variables taken together at all points for which

$$\begin{aligned} x &= 0,\ 1/k!,\ 2/k!,\ 3/k!,\ \ldots\ldots\ k!/k!, \\ y &= 0,\ 1/k!,\ 2/k!,\ 3/k!,\ \ldots\ldots\ k!/k!, \\ z &= 0,\ 1/k!,\ 2/k!,\ 3/k!,\ \ldots\ldots\ k!/k!, \end{aligned}$$

while it is continuous in (x, y, z) taken together at all other points. Now form

$$f(x, y, z) = \sum_{n=1}^{\infty} 1/n!\,\phi_n(x, y, z).$$

Since $\phi_n(x, y, z) = 1/3$ for all values of n, we have

$$\sum_{n=1}^{\infty} 1/n!\,\phi_n(x, y, z) = 1/3 \sum_{n=1}^{\infty} 1/n!,$$

and hence $f(x, y, z)$ converges. Now consider z as constant. Then $f(x, y, z)$, considered as a function of x, y, and n, converges uniformly by theorem 5 of section 10. In a similar manner $f(x, y, z)$ converges uniformly as a function of z and n if we consider x and y as constants. Hence $f(x, y, z)$ is continuous in (x, y) together and in z alone. But $f(x, y, z)$ is discontinuous in all three variables taken together in every subregion, for the general term $1/n!\phi_n(x, y, z)$ is discontinuous in (x, y, z) together at every point for which

$$x = p/q,\ y = p'/q',\ z = p''/q'',$$

where the p's and q's are positive integers, such that p and q are relatively prime to each other; likewise p' and q', and p'' and q''. In order that this shall be true, we need only choose n so large that $n!$ is divisible by q, q', and q''. Then $1/n!\phi_n(x, y, z)$ will have an (x, y, z) discontinuity at the point $x = p/q$, $y = p'/q'$, $z = p''/q''$, and hence $f(x, y, z)$ also has an (x, y, z) discontinuity at this point. This shows us that $f(x, y, z)$ has an (x, y, z) discontinuity at every rational point of our region, that is to say, at a set of points that is everywhere dense, while at the same time it is continuous in all three variables taken together at all other points. This proves the first part of the theorem.

The truth of the second part of the theorem follows at once from theorem 3. For if the function could have an (x, y, z) discontinuity greater than or equal to some positive number $\bar{k}$ at a set of points that is everywhere dense, then it would also have such a discontinuity at the limiting points, hence at every point of the region, which is impossible.

Corollary.—If a function $f(x, y, z)$ is continuous in each variable alone, then it may be discontinuous in all three variables taken together at a set of points that is everywhere dense, but the amount of the (x, y, z) discontinuity can in this case not be greater than or equal to some positive number $\bar{k}$ at each of these points.

Theorem 5.—If throughout a given region, $\alpha < x < \beta$, $a < y < b$, $z_0 < z < z_n$, the function $f(x, y, z)$ is continuous in two variables taken together and in the third variable alone, then it is continuous in all three variables taken together at a set of points that is at least everywhere dense in the region.

In order to prove this we choose a set of positive quantities

$$\bar{k}_1 > \bar{k}_2 > \bar{k}_3 > \dots\dots\dots\dots > \bar{k}_n > \dots\dots$$

where

$$\lim_{n=\infty} \bar{k} = 0.$$

Now consider that portion of our region which is confined within any sphere R lying entirely within the given region. Then it follows from theorem 4 that the points at which $f(x, y, z)$ has an (x, y, z) discontinuity greater than or equal to $\bar{k}_1$ cannot be everywhere dense in any part of this sphere. We can therefore find a sphere R_1 of radius r_1, lying wholly within the sphere R, such that within R_1 there are no points at which the (x, y, z) discontinuity of $f(x, y, z)$ is greater than or equal to $\bar{k}_1$. Again we can find a sphere R_2 of radius r_2, lying wholly within the sphere R_1, such that within the sphere R_2 there are no points at which the (x, y, z) discontinuity of the function is greater than or equal to $\bar{k}_2$. Repeating this reasoning we finally obtain a sphere R_n of radius r_n lying wholly within the sphere R_{n-1} and hence within every preceding sphere, such that within this sphere R_n there are no points at which the (x, y, z) discontinuity of the function is greater than or equal to $\bar{k}_n$. But we have by assumption that $\lim\limits_{n=\infty} r_n = 0$, and we see from the manner in which the spheres were obtained that the centers of the spheres approach a definite limiting point (x_0, y_0, z_0), at which therefore we have $k = 0$; that is to say, the function $f(x, y, z)$ is continuous in (x, y, z) taken together at this point. From the manner in which the spheres were chosen it follows that the points at which the limit of the (x, y, z) discontinuity is zero must exist in every part of our region, however small. They are therefore at least everywhere dense in the region.

We can, however, go a step further than this. Schoenfliess[*] shows namely that the set of points in question must have the "*Mächtigkeit*" of the continuum. To do this, choose at pleasure a set of positive quantities

$$k_1 > k_2 > k_3 > \ldots\ldots > k_n > \ldots,$$

where $\lim\limits_{n=\infty} k_n = 0$. Now let P_1 be a set of points within the region at which the (x, y, z) discontinuity of the function is greater than or equal to k. Similarly, let P_2 be a set of points within the region at which the (x, y, z) discontinuity is greater than or equal to k_2, etc. In general let P_n be a set of points of the region at which the (x, y, z) discontinuity of the function is greater than or equal to k_n. It follows now that P_n is a set of points having the following properties. Each set P_n is in no subregion everywhere dense; each set contains all the points of every preceding set; the set given by $\lim\limits_{n=\infty} P_n$ is a set that may be everywhere dense in the region. Such a set is called by Baire[†] a set of points of the *first category*. Baire also shows that the set of points complementary to such a set is not a set of the first category. He calls it a set of the *second category* and proves that it has the "*Mächtigkeit*" of the continuum. Hence it follows that the points at which the (x, y, z) discontinuity of our function is greater than or equal to some positive number k' is a set of the first category. The set of points therefore for which $\lim_{n=\infty} k_n = 0$ is the complementary set of the above set and hence is a set of the second category. It therefore has the "*Mächtigkeit*" of the continuum and is non-enumerable.

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[*] Cf. Schoenfliess: loco citato, page 129.

[†] Cf. Baire: Annali di matematici (3), vol. 3, page 67 (1899).

8. *Uniform Convergence.*—When dealing with the behavior of a function $f(x, y, z)$ not only within a given region but also on the boundaries of the region, we meet with a new kind of convergence, which has been called uniform convergence in the case of a function of two real variables. It is defined as follows.

If a function $f(x, y)$ of two real variables is defined for all values of $\alpha \leqq x \leqq \beta$, $y_0 < y \leqq y_0 + \delta$, and if $\lim_{y=y_0} f(x, y) = f(x, y_0)$ for every x of the interval $\alpha \leqq x \leqq \beta$, then we say that $f(x, y)$ converges uniformly to $f(x, y_0)$ if, having chosen an arbitrarily small $\sigma > 0$, we can then find a $\delta > 0$, such that

$$|f(x, y) - f(x, y_0)| < \sigma$$

for every y of $y_0 < y \leqq y_0 + \delta$, and for every x of the interval $\alpha \leqq x \leqq \beta$. This is equivalent to saying that the simultaneous limit $\lim_{\substack{x=x_0\\ y=y_0}} f(x, y)$ must exist for every x of the interval $\alpha \leqq x \leqq \beta$, and must equal the value $f(x_0, y_0)$.

When we come to a function of three real variables we have two cases to consider; for the function may converge uniformly either with respect to one of the variables or with respect to two of the variables.

In the former of these two cases we must have a function of three variables defined for every point of the region

$$\alpha \leqq x \leqq \beta,\ a \leqq y \leqq b,\ z_0 < z \leqq z_n,$$

and we must also have $\lim_{z=z_0} f(x, y, z) = f(x, y, z_0)$ for every (x, y) of the region. We then say that $f(x, y, z)$ converges uniformly to $f(x, y, z_0)$, if the simultaneous limit $\lim_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals $f(x_0, y_0, z_0)$ at every point (x_0, y_0) of the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$. This definition may also be expressed otherwise as follows.

The function $f(x, y, z)$ shall be defined for every point of the given region and $\lim\limits_{z = z_0} f(x, y, z)$ shall equal $f(x, y, z_0)$ for every point of the region as defined for x and y. If now

1. We choose at pleasure a $\sigma > 0$ and can then find a $\delta > 0$ such that

$$|f(x, y, z) - f(x, y, z_0)| < \sigma$$

when $z_0 < z \leqq z_0 + \delta$, for every point (x, y) of $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, then we say that $f(x, y, z)$ converges uniformly to $f(x, y, z_0)$ throughout the region, boundaries included. Two things are essential in this definition. First, the same δ must fit every point (x, y) of the region, and, secondly, the region must include its boundaries.

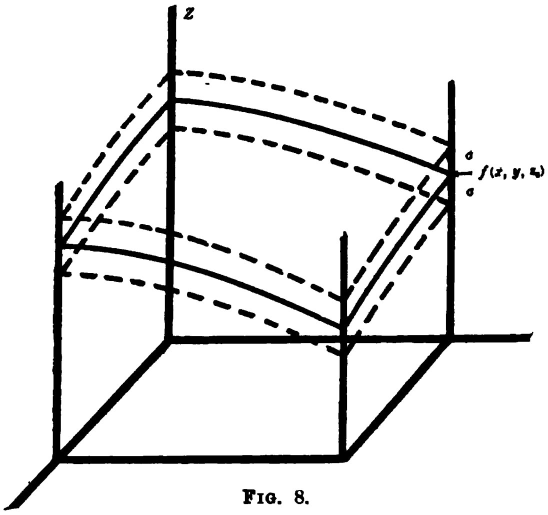

FIG. 8.

As in the case of a function of two variables, so here we have a geometrical interpretation of uniform convergence. The limiting function $f(x, y, z_0)$, considered as a function of x and y, may be represented by a surface. In a like manner we may represent the function $f(x, y, z_k)$, where $k = 1, 2, 3, \ldots\ldots$, by surfaces which we call the approximation surfaces. We now construct at a distance σ above and below $f(x, y, z_0)$ a surface parallel to $f(x, y, z_0)$, see figure 8.

If now we can find a z_k such that for every value of z, from z_k on, each of the approximation surfaces falls entirely within the strip bounded by the two parallel surfaces for every point (x, y) of the given region, then $f(x, y, z)$ converges uniformly to $f(x, y, z_0)$.

In the second case, where we have uniform convergence with respect to two variables, the function $f(x, y, z)$ must be defined for every value (x, y, z) of the region

$$\alpha \leqq x \leqq \beta,\ y_0 < y \leqq y_m,\ z_0 < z \leqq z_n$$

and the limit $\lim\limits_{\substack{y=y_0\\ z=z_0}} f(x, y, z)$ must equal $f(x, y_0, z_0)$ for every x of the interval $\alpha \leqq x \leqq \beta$. If under these conditions the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals $f(x_0, y_0, z_0)$ for every value of x_0 of its interval, then $f(x, y, z)$ converges uniformly to $f(x, y_0, z_0)$.

This can also be expressed otherwise. If $f(x, y, z)$ is defined throughout the given region and satisfies the condition that $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z) = f(x_0, y_0, z_0)$ for every x_0 of the interval $\alpha \leqq x \leqq \beta$, and if we choose an arbitrarily small $\sigma > 0$ and can then find a $\delta_1 > 0$ and a $\delta_2 > 0$, which are independent of each other and such that

$$|f(x, y, z) - f(x, y_0, z_0)| < \sigma$$

for every y and z where $y_0 < y \leqq y_0 + \delta_1$ and $z_0 < z \leqq z_0 + \delta_2$, then $f(x, y, z)$ converges uniformly to $f(x, y_0, z_0)$ throughont the interval $\alpha \leqq x \leqq \beta$.

Geometrically we can represent $f(x, y_0, z_0)$ by a curve in the interval $\alpha \leqq x \leqq \beta$. Similarly, $f(x, y_i, z_k)$, where $i = 1, 2, 3, \ldots.$ and $k = 1, 2, 3, \ldots.$ can be represented by a set of curves, which we call the approximation curves. Now draw at a distance σ above and below $f(x, y_0, z_0)$ curves parallel to the projections of the approximation curves. If then there exists a (y_i, z_k) such that, for every (y, z) from (y_i, z_k) on, the projections of the approximation curves all fall in the strip bounded by the two parallel curves, then $f(x, y, z)$ converges uniformly to $f(x, y_0, z_0)$.

The essential thing for uniform convergence, whether with respect to one variable or to two variables, is that the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals $f(x_0, y_0, z_0)$ for every point of the region, boundaries included. This fact is of vital importance. If, namely, we have a function defined as before, and if $\lim\limits_{\substack{y=y_0\\ z=z_0}} f(x, y, z) = f(x, y_0, z_0)$, and if we know that the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals $f(x_0, y_0, z_0)$ only for all points of the interval $\alpha < x < \beta$, where the endpoints are not included, then we do not know that the simultaneous limit also exists at the endpoints, and hence we cannot say that the function $f(x, y, z)$ converges uniformly throughout the interval endpoints included. All we can in this case say is that $f(x, y, z)$ converges uniformly throughout the interval $\alpha + \delta \leqq x \leqq \beta - \delta$, where δ is some positive number different from zero. The same considerations hold for uniform convergence with respect to two variables. The following example illustrates this.

Example 1.

Let $f(x, y, z) = \dfrac{xyz}{x^3 + y^3 + z^3}$, where $f(0, 0, 0) = 0$, for the region

$$0 \leqq x \leqq 1, \quad 0 \leqq y \leqq 1, \quad 0 < z \leqq 1.$$

This function is continuous in two variables taken together and in the third variable alone as has already been shown. Moreover the simultaneous limit $\lim\limits_{\substack{x=0\\ y=0\\ z=0}} f(x, y, z)$ exists and equals $f(0, 0, 0)$ for every (x, y) of the region $0 < x \leqq 1$, $0 < y \leqq 1$, but it does not exist for every (x, y) of the region when we include the boundaries. Hence the function does not converge uniformly throughout the region, boundaries included. In order to see what this means geometrically, let us plot a few of the approximation surfaces. We have the following table of corresponding values:

x	y	z	$f(x, y, z)$
0	0	1	0
1	1	1	1/3
1	1/2	1	4/17
1/2	1/2	1	1/5
1	1	1/2	4/17
1	1/2	1/2	1/5
1/2	1/2	1/2	1/3
1	1	1/3	9/55
1	1/2	1/3	36/251
1	1/3	1/3	3/29
1/2	1/2	1/3	9/31
1/3	1/3	1/3	1/3

This gives us the following approximation surfaces for $z = 1$, $z = 1/2$, $z = 1/3$, see figure 9.

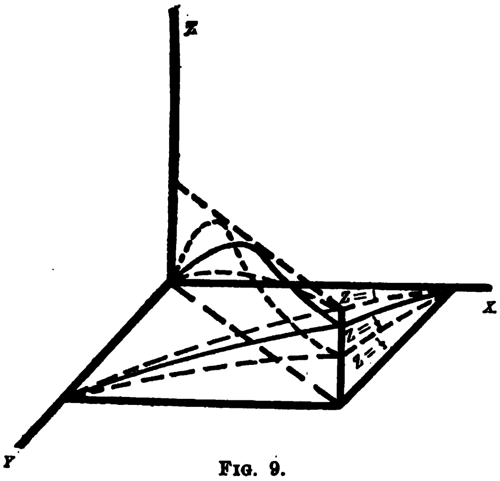

FIG. 9.

We see from this that each approximation surface has a peak of altitude 1/3 which moves along the line $y = x$ toward the origin as z approaches zero. At any point of the region, excepting the origin, we can therefore make $f(x, y, z)$ approach as closely as we please to zero by a suitable choice of z, but at the origin we cannot do so, and hence the function does not satisfy the condition of uniform convergence.

Having thus defined the two kinds of uniform convergence for a function of three variables, the question arises as to whether one of these will necessitate the other. That is, if $f(x, y, z)$ converges uniformly with respect to one variable, will it then also converge uniformly with respect to two variables, and conversely? This must be answered in the affirmative; for if $f(x, y, z)$ converges uniformly with respect to z, then we have

$$\lim_{z=z_0} f(x, y, z) = f(x, y, z_0) \tag{1}$$

for every (x, y) of the region, boundaries included. We also have

$$\lim_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z) = f(x_0, y_0, z_0) \tag{2}$$

at every point (x_0, y_0, z_0) of the region, boundaries included. From this last, however, follows that

$$\lim_{\substack{x=x_0\\ y=y_0}} f(x, y, z) = f(x_0, y_0, z) \tag{3}$$

for every z of its interval. Now (1), (2), and (3) are exactly the conditions that $f(x, y, z)$ shall converge uniformly with respect to the two variables x and y. Hence it follows that if $f(x, y, z)$ converges uniformly with respect to one variable, then it will also converge uniformly with respect to two variables. The converse is proved in a similar manner.

We have given two different definitions of uniform convergence and have asserted that they are equivalent to each other. We shall now prove that this is in fact the case.

Theorem 1.—If the function $f(x, y, z)$ is continuous in two variables taken together and in the third variable alone, and if we have $\lim\limits_{z=z_0} f(x, y, z) = f(x, y, z_0)$ for every (x, y) of $\alpha \leqq x \leqq \beta$ and $a \leqq y \leqq b$, then the necessary and sufficient condition that $f(x, y, z)$ converges uniformly to $f(x, y, z_0)$ throughout the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, is that at every point of this region, boundaries included, the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ shall exist and equal $f(x_0, y_0, z_0)$.

The condition is necessary; for let (x_0, y_0) be any point of the region and choose an arbitrarily small $\sigma > 0$. Now since $f(x, y, z)$ converges uniformly throughout the region, there exists a $\theta_0 > 0$, such that

$$|f(x, y, z_k) - f(x, y, z_0)| < 1/4\,\sigma \tag{1}$$

for every z_k between z_0 and $z_0 + \theta_0$ and for every point (x, y) of the region boundaries included. Furthermore, since $f(x, y, z)$ is continuous in (x, y) together, we can in the plane $z = z_k$ construct a circle of radius r_0 about (x_0, y_0), such that for every point (x, y, z_k) of this circle we have

$$|f(x_0, y_0, z_k) - f(x, y, z_k)| < 1/4\,\sigma. \tag{2}$$

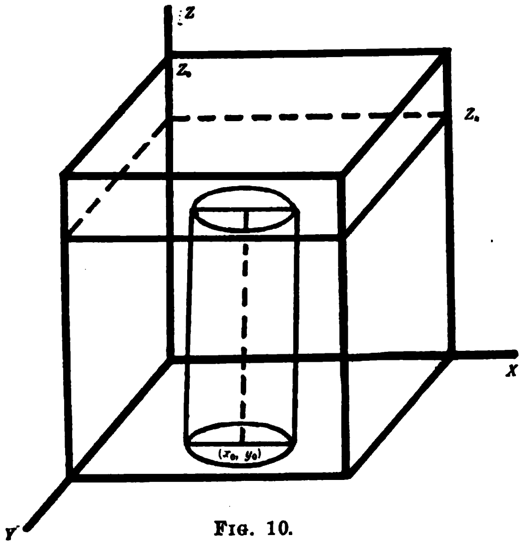

FIG. 10.

Now construct a right cylinder having this circle as base and its elements parallel to the z-axis, the bases of the cylinder lying in the planes $z = z_0$ and $z = z_k$, see figure 10.

Let (x, y, z) be any point within this cylinder. Since $f(x, y, z)$ is continuous in z alone we have by taking z_k near enough to z_0

$$|f(x, y, z_0) - f(x, y, z)| < 1/4\,\sigma \tag{3}$$

and

$$|f(x_0, y_0, z_0) - f(x_0, y_0, z_k)| < 1/4\,\sigma. \tag{4}$$

Adding these four inequalities we have

$$|f(x_0, y_0, z_0) - f(x, y, z)| < \sigma,$$

where (x, y, z) is any point within the cylinder. Hence the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists at (x_0, y_0, z_0) and equals $f(x_0, y_0, z_0)$. But (x_0, y_0) is any point of the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, and hence the simultaneous limit exists at every point of the region.

The condition is also sufficient; for since at any point (x_0, y_0, z_0) of our region, boundaries included, the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals $f(x_0, y_0, z_0)$, we can first choose an arbitrarily small $\sigma > 0$ and can then construct a sphere R_0 of radius r_0 about (x_0, y_0, z_0) as center, such that there shall exist no point (ξ, η, ζ) within this sphere for which

$$|f(x_0, y_0, z_0) - f(\xi, \eta, \zeta)| \leqq \sigma. \tag{5}$$

For different points (x, y) of the region, but for the same value of σ, the radii of the resulting spheres may vary and are functions of (x, y). As such they have a lower limit ρ' different from zero, for the entire region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$; for suppose that these radii approach zero as limit as (x, y) approaches some point (x_0, y_0) of the region $(\alpha, \beta; a, b)$. Then it would be impossible to find any region about (x_0, y_0, z_0) which is free from points (ξ, η, ζ) that satisfy (5). Furthermore, the limiting point (x_0, y_0, z_0) must also be a point of the region. The radii ρ therefore have a definite lower limit ρ' on on the region $(\alpha, \beta; a, b)$ and this may therefore serve for the whole region, so that the definition for uniform convergence is satisfied.

9. *Continuity of the Limiting Function.*—Thus far we have not said anything about the continuity of the limiting functions $f(x, y, z_0)$ and $f(x, y_0, z_0)$. We now take up theorems that deal with this.

Theorem 1.—If the function $f(x, y, z)$ converges uniformly with respect to z, that is, if it converges uniformly to $f(x, y, z_0)$ throughout the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, then $f(x, y, z_0)$ is continuous in (x, y) together throughout this region.

Let (x_0, y_0) be any point of the given region. Since $f(x, y, z)$ converges uniformly throughout the region, the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists for every point of the region, boundaries included, and equals $f(x_0, y_0, z_0)$. But we also have $\lim\limits_{z=z_0} f(x, y, z) = f(x, y, z_0)$ for every (x, y) in the neighborhood of (x_0, y_0). Hence we have by theorem 4 of section 2

$$\lim_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z) = \lim_{\substack{x=x_0\\ y=y_0}} \lim_{z=z_0} f(x, y, z) = \lim_{z=z_0} \lim_{\substack{x=x_0\\ y=y_0}} f(x, y, z) = f(x_0, y_0, z_0).$$

From this follows at once that $f(x, y, z_0)$ is continuous in (x, y) together at (x_0, y_0). But since (x_0, y_0) is any point of the region, boundaries included, it follows that $f(x, y, z_0)$ is continuous in (x, y) throughout the entire region.

Theorem 2. If the function $f(x, y, z)$ converges uniformly with respect to y and z, that is if it converges uniformly to $f(x, y_0, z_0)$ in the interval $\alpha \leqq x \leqq \beta$, then $f(x, y_0, z_0)$ is continuous in x throughout the interval including the boundaries.

The proof is similar to that of theorem 1. Let x_0 be any point of the interval $\alpha \leqq x \leqq \beta$. Since $f(x, y, z)$ converges uniformly to $f(x, y_0, z_0)$ throughout the interval, the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ must exist at every point of the interval, end points included, and must equal $f(x_0, y_0, z_0)$. But we have also $\lim\limits_{\substack{y=y_0\\ z=z_0}} f(x, y, z) = f(x, y_0, z_0)$ for every x of the interval. Hence we have by theorem 3 of section 2

$$\lim_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z) = \lim_{x=x_0} \lim_{\substack{y=y_0\\ z=z_0}} f(x, y, z) = \lim_{\substack{y=y_0\\ z=z_0}} \lim_{x=x_0} f(x, y, z) = f(x_0, y_0, z_0).$$

From this follows at once that $f(x, y_0, z_0)$ is continuous in x at the point $x = x_0$. But x_0 is any point of the interval $\alpha \leqq x \leqq \beta$, and hence $f(x, y_0, z_0)$ must be continuous in x throughout the interval, end points included.

Remark.—Theorems 1 and 2 give us sufficient but not necessary conditions for the continuity of the limiting functions $f(x, y, z_0)$ and $f(x, y_0, z_0)$. We can show this by means of an example which has already been used several times.

Let $f(x, y, z) = \dfrac{xyz}{x^3 + y^3 + z^3}$, where $f(0, 0, 0) = 0$. Here $f(x, y, 0) = 0$

throughout the region $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, and it is therefore continuous in (x, y) together at the point $(0, 0)$. Likewise $f(x, 0, 0) = 0$ throughout the interval $0 \leqq x \leqq 1$, and it is therefore continuous in x at the point $x = 0$. But the function $f(x, y, z)$ does not converge uniformly either to $f(x, y, 0)$ or to $f(x, 0, 0)$ as has already been shown.

It is, however, possible to obtain a necessary and sufficient condition that the limiting function $f(x, y, z_0)$ is continuous in (x, y) together at any point (x, y) of the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, and that the limiting function $f(x, y_0, z_0)$ is continuous in x at any point x of the interval $\alpha \leqq x \leqq \beta$.

Theorem 3.*—If the function $f(x, y, z)$ is finite and continuous in (x, y) together and in z alone throughout the region

$$\alpha \leqq x \leqq \beta,\ a \leqq y \leqq b,\ z_0 < z \leqq z_n,$$

then the necessary and sufficient condition that the limiting function $f(x, y, z_0)$ is finite and also continuous in (x, y) together at any fixed point (x_0, y_0) of the region

$$\alpha \leqq x \leqq \beta,\ a \leqq y \leqq b, \tag{A}$$

is

(1) that $\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$ for every constant pair of values (x, y) of region (A).

(2) that having chosen an arbitrarily small $\sigma > 0$, there shall then exist a set of z's ($z = z_1, z_2, z_3, \ldots$) dense at $z = z_0$, and that corresponding to each z_k there shall exist a circle of radius r_k about (x_0, y_0), where however r_k may vary with z_k, such that for all points (x, y) within this circle

$$|f(x, y, z_k) - f(x, y, z_0)| < \sigma.$$

We first prove that the condition is necessary. Our hypothesis therefore is that $f(x, y, z_0)$ is finite and continuous in (x, y) together at (x_0, y_0) where (x_0, y_0) is some fixed point of region (A). Hence we can choose an arbitrarily small $\sigma > 0$ and can then, in the plane $z = z_0$, find a circle of radius r_0 about (x_0, y_0), such that for every point (x, y) of this circle we have

$$|f(x, y, z_0) - f(x_0, y_0, z_0)| < 1/3\,\sigma \tag{1}$$

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* Arzela proves this theorem for the case of a function of two variables. See "Sulle serie di funzioni." Accademia delle sc. di Bologna, 1899, page 142.

By hypothesis we also have $\lim_{z=z_0} f(x, y, z) = f(x, y, z_0)$, and hence also $\lim_{z_k=z_0} f(x, y, z_k) = f(x, y, z_0)$, for every (x, y) of region (A). From this follows that for the same σ as above we can determine a $\delta > 0$, such that

$$|f(x_0, y_0, z_k) - f(x_0, y_0, z_0)| < 1/3\,\sigma \tag{2}$$

where $$z_0 < z \leqq z_0 + \delta.$$

We also have that $f(x, y, z)$ is continuous in (x, y) together, when z is constant, at every point of region (A), and hence we can in the plane $z = z_k$ determine a circle of radius r_k about (x_0, y_0), such that for every point of this circle

$$|f(x, y, z_k) - f(x_0, y_0, z_k)| < 1/3\,\sigma \tag{3}$$

where $$z_0 < z \leqq z_0 + \delta.$$

Inequalities (1), (2), and (3) hold for all values of z and z_k between z_0 and $z_0 + \delta$. If $r_k < r_0$, then (3) holds for all points (x, y) of a circle of radius r_k about (x_0, y_0) in the plane $z = z_k$. We may then add these inequalities and obtain

$$|f(x, y, z_k) - f(x, y, z_0)| < \sigma,$$

and hence the condition is necessary.

We next prove that the condition is also sufficient. By hypothesis we now have that $\lim_{z=z_0} f(x, y, z) = f(x, y, z_0)$ for every point (x, y) of region (A). Hence we can choose at pleasure $\sigma > 0$ and can then determine $\delta > 0$, such that

$$|f(x_0, y_0, z_s) - f(x_0, y_0, z_0)| < 1/4\,\sigma \tag{4}$$

where $$z_0 < z_s \leqq z_0 + \delta.$$

But $f(x, y, z)$ is continuous in (x, y) together at every point of region (A), and hence we can for the same σ as above find, in the plane $z = z_s$, a circle of radius r_s about (x_0, y_0), such that for all points of this circle

$$|f(x, y, z_s) - f(x_0, y_0, z_s)| < 1/4\,\sigma \tag{5}$$

where $$z_0 < z_s \leqq z_0 + \delta.$$ (See figure 11.)

From the second part of the hypothesis follows that in the plane $z = z_k$ there exists a circle of radius $\bar{r}_k$ about (x_0, y_0), such that for all points of this circle

$$|f(x, y, z_0) - f(x, y, z_k)| < 1/4\,\sigma \tag{6}$$

where $$z_0 < z_k \leqq z_0 + \delta.$$

It is to be noticed that if $z_s = z_k$, then it is not necessary that also $r_s = \bar{r}_k$, for the two circles do not arise from the same conditions. We may however choose r_s to fit both cases if $r_s < \bar{r}_k$. Since $\lim\limits_{z_k = z_0} f(x, y, z_k) = f(x, y, z_0)$ for every constant pair of values (x, y) of region (A), we have for the same σ as above

$$|f(x_0, y_0, z_k) - f(x_0, y_0, z_0)| < 1/4\,\sigma \tag{7}$$

where

$$z_0 < z_k \leqq z_0 + \delta.$$

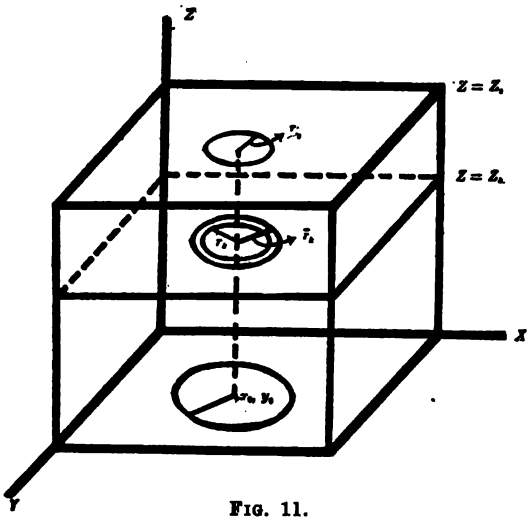

FIG. 11.

The continuity of $f(x, y, z)$ in (x, y) together throughout region (A) also gives us

$$|f(x, y, z_k) - f(x_0, y_0, z_k)| < 1/4\,\sigma \tag{8}$$

where

$$z_0 < z_k \leqq z_0 + \delta,$$

for every point (x, y) of the circle of radius $\bar{r}_k$ about (x_0, y_0) in the plane $z = z_k$. We will suppose that $r_s < \bar{r}_k$, so that inequality (8) holds for all points of the circle of radius r_s. Inequalities 4, 5, 7, and 8, now hold for every z where $z_0 < z_k \leqq z_0 + \delta$, and for all points (x, y) of the circle of radius r_s. We may therefore add them and obtain

$$|f(x, y, z_s) - f(x, y, z_k)| < \sigma \tag{9}$$

where

$$z_0 < z_s \leqq z_0 + \delta,$$

and for all points (x, y) of a circle of radius r_s about (x_0, y_0). We can now choose

δ in such a manner that 4, 5, and 9, shall hold for every z where $z_0 < z_s \leqq z_0 + \delta$, so that we may add these inequalities and thus obtain

$$|f(x, y, z_0) - f(x_0, y_0, z_0)| < 3/2\,\sigma \tag{10}$$

where (x, y) is any point within the circle of radius r_s. Hence there exists in the plane $z = z_0$ about (x_0, y_0) a circle of radius greater than zero, such that (10) holds at every point within this circle. Hence $f(x, y, z_0)$ is continuous in (x, y) together at (x_0, y_0). This proves the theorem.

We can obtain an analogous theorem for the continuity of the limiting function $f(x, y_0, z_0)$ at the point $x = x_0$.

Theorem 4. If the function $f(x, y, z)$ is finite and continuous in x alone and in (y, z) together at every point of the region

$$\alpha \leqq x \leqq \beta, \quad y_0 < y \leqq y_m, \quad z_0 < z \leqq z_n,$$

then the necessary and sufficient condition that the limiting function $f(x, y_0, z_0)$ is finite and continuous in x at x_0 where x_0 is some fixed point of the interval

$$\alpha \leqq x \leqq \beta, \tag{A}$$

is

(1) that $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$ for every x of the interval (A);

(2) that having chosen an arbitrarily small $\sigma > 0$, there shall then exist a set of pairs of values (y_i, z_k) which shall be dense at (y_0, z_0) and corresponding to each pair there shall exist an interval greater than zero, such that for every x of this interval

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma.$$

The proof of this theorem is similar to that of the preceding theorem. Since $f(x, y_0, z_0)$ is continuous in x at the point $x = x_0$, we can choose an arbitrarily small $\sigma > 0$ and can then determine an interval $(x_0 - \delta_1, x_0 + \delta_2)$ greater than zero, such that

$$|f(x_0, y_0, z_0) - f(x, y_0, z_0)| < 1/3\,\sigma \tag{1}$$

for every x of this interval. Since $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$ and hence also

$\lim\limits_{\substack{y_i = y_0 \\ z_k = z_0}} f(x, y_i, z_k) = f(x, y_0, z_0)$ for every x of the interval (A), we can for the same σ find some value (y_s, z_t), such that

$$|f(x_0, y_i, z_k) - f(x_0, y_0, z_0)| < 1/3\,\sigma \tag{2}$$

where $\qquad y_0 < y_i \leqq y_s, \quad z_0 < z_k \leqq z_t.$

But $f(x, y, z)$ is continuous in x when y and z are constant, hence for the same σ we can determine a $\bar{\delta}_1$ and a $\bar{\delta}_2$ each greater than zero, such that

$$|f(x, y_i, z_k) - f(x_0, y_i, z_k)| < 1/3\,\sigma \tag{3}$$

where $\qquad x_0 - \bar{\delta}_1 < x < x_0 + \bar{\delta}_2.$

Now let δ_1 and δ_2 be less than $\bar{\delta}_1$ and $\bar{\delta}_2$ respectively. Then we may add (1), (2), and (3), and thus obtain

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma$$

where $\qquad x_0 - \delta_1 < x < x_0 + \delta_2.$

This proves that the condition is necessary.

The condition is also sufficient. We now have by hypothesis that $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$ for every x of the interval (A). Hence we can choose at pleasure a $\sigma > 0$ and can then find a δ_1 and a δ_2 each greater than zero, such that

$$|f(x_0, y_0, z_0) - f(x_0, y_s, z_t)| < 1/6\,\sigma \tag{4}$$

where $\qquad y_0 < y_s < y_0 + \delta_1$ and $z_0 < z_t < z_0 + \delta_2.$

Since $f(x, y, z)$ is continuous in x, we can for the same σ and on the line $(x = y_s,\ z = z_t)$ find about x an interval $(x_0 - \delta_s,\ x_0 + \delta_s)$, such that

$$|f(x_0, y_s, z_t) - f(x, y_s, z_t)| < 1/6\,\sigma \tag{5}$$

where $\qquad x_0 - \delta_s < x < x_0 + \delta_s.$

From the second part of the hypothesis it follows that on the line $(y = y_i,\ z = z_k)$ we can for the same value of σ determine an interval $(x_0 - \delta_k,\ x_0 + \delta_k)$, such that

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < 1/6\,\sigma \tag{6}$$

where $\qquad x_0 - \delta_k < x < x_0 + \delta_k, \quad y_0 < y_i \leqq y_0 + \delta_1, \quad z_0 < z_k \leqq z_0 + \delta_2.$

If $y_i = y_s$ and $z_k = z_t$, then it is not necessary that also $\delta_s = \delta_k$. But if $\delta_s < \delta_k$, then δ_s will fit both cases. Let us choose it thus. Since $\lim\limits_{\substack{y_i = y_0 \\ z_k = z_0}} f(x, y_i, z_k) = f(x, y_0, z_0)$ for every x of the interval (A), we can for the same value of σ determine a $\bar{\delta}_1$ and a $\bar{\delta}_2$, such that

$$|f(x_0, y_i, z_k) - f(x_0, y_0, z_0)| < 1/6\,\sigma \tag{7}$$

where $$y_0 < y_i \leqq y_0 + \bar{\delta}_1, \quad z_0 < z_k \leqq z_0 + \bar{\delta}_2.$$

Finally we have from the continuity of $f(x, y, z)$ in x that for the same σ we can find a $\bar{\delta}_k > 0$, such that

$$|f(x, y_i, z_k) - f(x_0, y_i, z_k)| < 1/6\,\sigma \tag{8}$$

where $$x_0 - \bar{\delta}_k < x < x_0 + \bar{\delta}_k.$$

Now let δ_s be less than δ_k, and let δ_1 and δ_2 be less than $\bar{\delta}_1$ and $\bar{\delta}_2$ respectively. Then (4), (5), (7), and (8), hold for the same intervals and we may add them. We thus obtain

$$|f(x, y_s, z_t) - f(x, y_i, z_k)| < 2/3\,\sigma \tag{9}$$

where $y_0 < y \leqq y_0 + \delta_1$ and $z_0 < z \leqq z_0 + \delta_2$, and for all points x of the interval $(x_0 - \delta_s,\ x_0 + \delta_s)$. We can now choose δ_1 and δ_2 in such a manner that the inequalities (4), (5), and (9), hold simultaneously, so that we may add these and thus obtain

$$|f(x, y_0, z_0) - f(x_0, y_0, z_0)| < \sigma \tag{10}$$

where x is any point of the interval $(x_0 - \delta,\ x_0 + \delta)$. Hence on the line $(y = y_0,\ z = z_0)$ there exists about x_0 an interval greater than zero for every point of which inequality (10) holds. Hence $f(x, y_0, z_0)$ is continuous in x at the point $x = x_0$, which proves the theorem.

We have thus obtained necessary and sufficient conditions that the functions $f(x, y, z_0)$ and $f(x, y_0, z_0)$ are continuous in (x, y) and x respectively at some fixed point. We next wish to find the condition that they shall be continuous throughout a whole region or interval. Such a condition is that the conditions of theorems 3 and 4 hold for every point of the region or interval in question. We can, however, obtain another condition which will be of use later on.

Before giving this condition we must first define another kind of uniform convergence. This is Arzela's "*Convergenza uniforme a tratti and a stratti.*" *

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* See Arzela: Sulle serie di funzioni. Accademia delle scienze di Bologna, 1899, p. 153.

We may translate this by uniform convergence by segments and by areas. We first give the definition for a function of two variables.

The function $f(x, y)$ shall be defined for all points of the region $\alpha \leqq x \leqq \beta$, $y_0 < y \leqq y_n$, and shall be continuous in each variable alone throughout this region; furthermore $\lim\limits_{y=y_0} f(x, y)$ shall equal $f(x, y_0)$ for every x for which it is defined. If now there exists between y_0 and y_n a finite number of lines ($y = y_1, y_2, y_3, \ldots, y_p$) and on each of these a finite number of segments each greater than or equal to some finite length L which is itself greater than zero, such that the projections of these segments on $y = y_0$ completely fill the interval (α, β) and that for every point of these segments

$$|f(x, y_0) - f(x, y_k)| < \sigma,$$

then we say that $f(x, y)$ converges uniformly by segments to $f(x, y_0)$ throughout the interval.

In an analogous manner we define uniform convergence by areas for a function of three variables. Let the function $f(x, y, z)$ be defined for every point of the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, $z_0 < z \leqq z_n$, and let it be continuous in (x, y), together and in z alone throughout the region; furthermore let $\lim\limits_{z=z_0} f(x, y, z) = f(x, y, z_0)$ for every pair of values (x, y) of the region. If now there exists between z_0 and z_k a finite number of planes ($z = z_1, z_2, z_3, \ldots, z_p$) and on each of these planes a finite number of rectangles having their sides parallel to the x and y axes and such that their dimensions are each greater than or equal to some quantity greater than zero, and furthermore such that the projections of these rectangles on the plane $z = z_0$ completely fill the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, and that for all points (x, y) of these rectangles the following inequality holds

$$|f(x, y, z_0) - f(x, y, z_k)| < \sigma,$$

then we say that $f(x, y, z)$ converges uniformly by areas to $f(x, y, z_0)$ throughout the region.

We may also define such uniform convergence for the case that $f(x, y, z)$ converges to $f(x, y_0, z_0)$. Let $f(x, y, z)$ be defined for every point of the region $\alpha \leqq x \leqq \beta$, $y_0 < y \leqq y_m$, $z_0 < z \leqq z_n$, and let it be continuous throughout this region in (y, z) together and in x alone. Furthermore let $\lim\limits_{\substack{y=y_0\\ z=z_0}} f(x, y, z) = f(x, y_0, z_0)$

for every constant value x of the interval. If now we have a finite set of values (y_i, z_k), where the absolute value of $\sqrt{(y_i^2 + z_k^2)}$ for each (y_i, z_k) is less than the corresponding value for the preceding (y_i, z_k), and if on each line $(y = y_i, z = z_k)$ there exists a finite number of segments, each of length greater than or equal to L, where L is itself greater than zero, such that the projections of these segments on the line $(y = y_0, z = z_0)$ completely fill the interval (α, β) and that for every point x of these segments we have

$$|f(x, y_0, z_0) - f(x, y_i, z_k)| < \sigma,$$

then we say that $f(x, y, z)$ converges uniformly by segments to $f(x, y_0, z_0)$ throughout the interval.

This new kind of uniform convergence is entirely distinct from both the ordinary uniform convergence and from Dini's "*einfach gleichmässige Convergenz*", as Arzela[*] has shown in the case of a function of two variables. For a function of three variables the distinction between these three kinds of uniform convergence is as follows.

In all three cases the function $f(x, y, z)$ is defined for all points of the region

$$\alpha \leqq x \leqq \beta, \; a \leqq y \leqq b, \; z_0 < z \leqq z_n,$$

and $\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$ for every point of the region

$$\alpha \leqq x \leqq \beta, \; a \leqq y \leqq b. \tag{A}$$

We then choose at pleasure a $\sigma > 0$.

In the case of ordinary uniform convergence we must then be able to find a $z_s \neq z_0$, such that for *every* z between z_s and z_0 and for every point (x, y) of region (A) we have

$$|f(x, y, z) - f(x, y, z_0)| < \sigma. \tag{1}$$

In the case of Dini's "einfach gleichmässigs Convergenz" there exists between z_s and z_0 only a set of z*'s* dense at z_0, such that (1) holds for each of these z*'s* and for every (x, y) of region (A).

In the case of Arzela's uniform convergence by areas there exists between z_s and z_0 only a finite number of planes $z = z_k$, and on each such plane a finite number of rectangles, each of area greater than or equal to G where G is finite and greater than zero, whose projections on the plane $z = z_0$ completely fill region (A), such that (1) holds only for all points of these rectangles.

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[*] Arzela: loco citato, page 158.

We are now ready to take up the theorems that deal with the continuity of the limiting function throughout the whole region.

Theorem 5.[*] If the function $f(x, y, z)$ is continuous in (x, y) together and in z alone throughout the region

$$a \leqq x \leqq \beta,\ a \leqq y \leqq b,\ z_0 < z \leqq z_n,$$

and if $\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$ for every (x, y) of the region

$$a \leqq x \leqq \beta,\ a \leqq y \leqq b, \tag{A}$$

then the necessary and sufficient condition that $f(x, y, z_0)$ is continuous in (x, y) together throughout region (A) is that $f(x, y, z)$ shall converge uniformly by areas to $f(x, y, z_0)$ throughout (A).

The condition is necessary. Then our hypothesis is that $f(x, y, z_0)$ is continuous in (x, y) together at every point of region (A). If therefore (x', y') is any point of region (A), then it follows from theorem 3 of this section that there exists a set of z's $(z = z_1, z_2, z_3, \ldots.)$ dense at $z = z_0$, and on each z_k sufficiently near to z_0 there exists about (x', y') a rectangle having its sides parallel to the axes and having an area δ_k different from zero, such that for every point (x, y) within this rectangle

$$|f(x, y, z_k) - f(x, y, z_0)| < \sigma, \tag{3}$$

where σ is an arbitrarily small positive number. It will be advantageous in the following not to consider this whole rectangle about (x', y'), but to divide it into four rectangles by lines parallel to the axes and to consider each of these rectangles separately. Let us take, for example, the rectangle $P'ABC$ to the right and front of (x', y'). See fig. 12.

We may then say that, having chosen an arbitrarily small $\sigma > 0$ and any z_s, there will then exist for every z_k between z_s and z_0 a rectangle of area greater than zero to the right and front of (x', y'), such that (1) holds for every point (x, y) of this rectangle. The area of this rectangle may vary with z_k and may in fact be zero for some values of z_k, but it cannot be zero for every value of z_k, unless (x', y') is a point of the boundaries $x = \beta$ and $y = b$. The area of this rectangle is moreover a function of z_k and as such it has an upper limit which is certainly less than the area of the region (A). Let us denote this upper limit by $\Delta(x', y')$. If now (x', y') is considered as a variable, then $\Delta(x', y')$ is a function of (x', y'). Consider now the region $a \leqq x \leqq \beta - \varepsilon'$, $a \leqq y \leqq b - \varepsilon''$, where ε' and ε'' are two positive numbers chosen at pleasure, excepting that they shall be less than the corresponding dimensions of region (A). We can now show that within this new region $\Delta(x', y')$ considered as a function of (x, y) has a lower limit different from zero. Let this lower limit be denoted by l. There exists some point (x, y) of the region $a \leqq x \leqq \beta - \varepsilon'$, $a \leqq y \leqq b - \varepsilon''$, such that the lower limit of $\Delta(x', y')$ is l in every neighborhood of this point (here $\Delta(x', y')$ is always the area of the rectangle to the right and front of the point).

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[*] Arzela: loco citato, page 147.

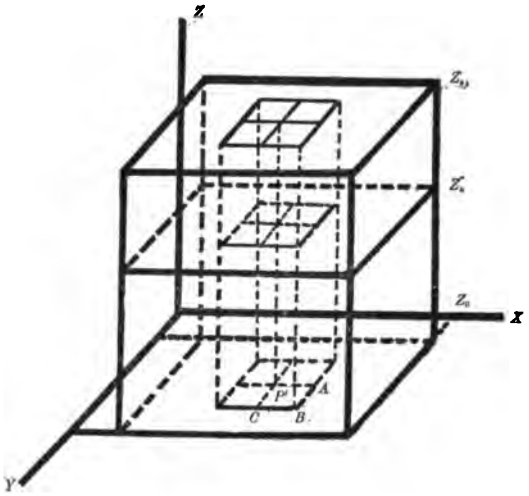

FIG. 12.

Let this point be (x_1, y_1). For this point the value of the function $\Delta(x_1, y_1)$ is certainly greater than zero. Now there surely exists among the values of z between z_s and z_0 one value z_k for which the area of the rectangle, for every point (x, y) of which we have

$$|f(x, y, z_k) - f(x, y, z_0)| < \sigma,$$

is as near as we please to $\Delta(x_1, y_1)$. It is now evident that, if we consider on $z = z_0$ a rectangle of area $1/4\Delta(x_1, y_1)$, then we shall have for every point (x, y) of this rectangle at least a rectangle of area $1/4\Delta(x_1, y_1)$ to the right and front, such that for every point (x, y) of this rectangle the above inequality holds. Hence l must be greater than or equal to $1/4\Delta(x_1, y_1)$. But since this is greater than zero it follows that l is greater than zero.

In the same way we can show that the upper limit of each of the other three rectangles, into which we divided the original rectangle about (x', y'), when considered as a function of (x, y) has a lower limit greater than zero. Hence the entire rectangle about (x', y'), which is the upper limit for a fixed (x, y), has a lower limit greater than zero when considered as a function of (x, y).

We have thus shown that this lower limit exists and is greater than zero for the region (A) exclusive of the boundaries. It remains to take care of the boundaries. In order to do this we consider the region

$$\alpha + \varepsilon' \leqq x \leqq \beta - \varepsilon',\ a + \varepsilon'' \leqq y \leqq b - \varepsilon''. \tag{B}$$

Let the lower limit of the area of the rectangle within the region (B) for every point of which

$$|f(x, y, z_k) - f(x, y, z_0)| < \sigma$$

be L. Now consider the remaining strip of region (A). It is evident that within this strip the dimensions of the rectangle for every point of which (1) holds cannot exceed ε' and ε''. Hence the lower limit L of the rectangle for the whole region (A) cannot exceed ε', ε''. Now let L' be some fixed positive quantity less than L. We can then say that, having chosen an arbitrarily small positive quantity σ and an arbitrary z_t, we can then find between z_t and z_0 a finite number of planes $z = z_k$, and on each a finite number of rectangles with their sides parallel to the axes, each rectangle having an area greater than or equal to L', such that for every point (x, y) of these rectangles

$$|f(x, y, z_k) - f(x, y, z_0)| < \sigma.$$

We thus prove that the condition of the theorem is necessary.

It remains to prove that it is also sufficient. Let us choose at pleasure a $\sigma > 0$ and any $z = z_t$. Let (x_0, y_0) be any point of region (A). Then it follows from the second part of the hypothesis that there exists between z_t and z_0 a finite number of planes $z = z_1, z_2, z_3, \ldots, z_p$, and on each a finite number of rectangles with sides parallel to the axes and each of area greater than zero, such that for all points (x, y) of these rectangles

$$|f(x, y, z) - f(x, y, z_0)| < \sigma. \tag{1}$$

Since the projections of these rectangles on $z = z_0$ completely fill region (A) by hypothesis, it follows that the line through (x_0, y_0) perpendicular to the XY-plane must meet at least one of these rectangles. Let this be the rectangle on the plane $z = \bar{z}_1$. If now we let σ remain unchanged and choose $\bar{z}_1$ as a new z_t, then there exists between $\bar{z}_1$, and z_0 a set of planes finite in number and on each such plane a finite number of rectangles each of area greater than zero, such that for all points of these rectangles inequality (1) holds. Since again the projections of these rectangles on the plane $z = z_0$ completely fill region (A), the perpendicular through (x_0, y_0) to the XY-plane must meet at least one of these rectangles. Let it be a rectangle in the plane $z = \bar{z}_2$. Again let σ remain unchanged and choose $\bar{z}_2$ as our now z_t and repeat the reasoning. We thus obtain between z_t and z_0 a set of z's $(z = \bar{z}_1, \bar{z}_2, \bar{z}_3, \dots)$ dense at $z = z_0$, and on each plane $z = \bar{z}_t$ a rectangle of area greater than zero about (x_0, y_0), such that for all points of these rectangles inequality (1) holds. Hence it follows from theorem 3 of this section that $f(x, y, z_0)$ is continuous in (x, y) together at the point (x_0, y_0). But (x_0, y_0) is any point of region (A) and therefore the function $f(x, y, z_0)$ is continuous in (x, y) together at every point of region (A). It is evident that if (x_0, y_0) had been a point of the boundary of region (A), then the rectangle in question could have been taken only up to the boundary.

We have an analogous theorem for the continuity of the limiting function $f(x, y_0, z_0)$. It is the following.

Theorem 6. If the function $f(x, y, z)$ is continuous in x alone and in (y, z) together throughout the region

$$\alpha \leqq x \leqq \beta\,,\ y_0 < y \leqq y_m,\ z_0 < z \leqq z_n\,,$$

and if $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$ for every x of the interval

$$\alpha \leqq x \leqq \beta\,, \tag{A}$$

then the necessary and sufficient condition that $f(x, y_0, z_0)$ is continuous in x throughout this interval is that $f(x, y, z)$ converges uniformly by segments to $f(x, y_0, z_0)$ throughout the interval.

The proof of this theorem is analogous to that of theorem 5. First the condition is necessary. Then we have by hypothesis that $f(x, y_0, z_0)$ is continuous in x throughout the interval (A) and we wish to prove that throughout this interval the function $f(x, y, z)$ converges uniformly by segments to $f(x, y_0, z_0)$. Since $f(x, y_0, z_0)$ is continuous in x throughout the interval (A), it follows from theorem 4 of this section that for any point x' of this interval there exists a set of values (y_i, z_k) dense at (y_0, z_0) and corresponding to each such pair of values there exists a $\delta_k > 0$, such that

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma \tag{1}$$

where $x' - \delta_k \leqq x \leqq x' + \delta_k$. It will be more accurate to consider separately the intervals to the left and to the right of x'. Having chosen an arbitrarily small positive number σ, we can then find a set of values (y_i, z_k) dense at (y_0, z_0) and corresponding to each pair (y_i, z_k) a positive δ_k, such that

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma$$

for every x of the interval of length δ_k to the right of x'. For every value (y_i, z_k) we obtain a corresponding δ_k. We choose in each case the largest possible value of the interval to the right of x', and it shall be this interval that is denoted by δ_k. It is then evident that δ_k cannot be zero for every pair of values (y_i, z_k) unless $x' = \beta$. Moreover δ_k is a function of (y_i, z_k) and as such it has an upper finite limit which is surely not greater than $\beta - \alpha$. Let us denote this upper limit by $\Delta(x')$. If now we let x' become variable, then $\Delta(x')$ is a function of x, which we shall consider in the interval $\alpha \leqq x \leqq \beta - \varepsilon$, where ε is any positive number less than $\beta - \alpha$. We can now assert that $\Delta(x')$ has a lower limit greater than zero when considered as a function of x in the interval $\alpha \leqq x \leqq \beta - \varepsilon$. Let l be the lower limit in question. Then there certainly exists some point x within the interval $(\alpha, \beta - \varepsilon)$, such that in every neighborhood of this point the lower limit of $\Delta(x')$ is l. Let x_1 be such a point. We then surely know that for $x = x_1$ the value $\Delta(x')$ of the interval to right must be greater than zero, since $x \neq \beta$. If therefore we consider the interval $x_1 \leqq x \leqq x_1 + 1/2\,\Delta(x_1)$, and if x'' is any point of this interval, then there certainly exists an interval of length at least equal to $1/2\,\Delta(x_1)$ to the right of x'', such that for every x of this interval.

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma.$$

Hence l must be at least equal to $1/2\,\Delta(x_1)$. This is however greater than zero and hence l is greater than zero. In a like manner we can show that in the interval $(\alpha + \varepsilon', \beta)$ there exists a similar lower limit l' greater than zero for the upper limits of $\Delta(x')$. It follows from this that for the interval $(\alpha + \varepsilon' \leqq x \leqq \beta - \varepsilon)$ there exists a lower limit L greater than zero for the total interval about x for which (1) holds.

From what has been said it follows that for the point $x = \beta$ there must exist a lower limit greater than zero of the intervals to the left. Call this Δ_β. Similarly there exists a lower limit greater than zero of the intervals to the right of the point $x = \alpha$. Call this Δ_α. Then there exists for the interval

$\alpha + \Delta_\alpha \leqq x \leqq \beta - \Delta_\beta$ a lower limit L greater than zero of the total interval about x for which (1) holds. For the whole interval $\alpha \leqq x \leqq \beta$ the lower limit of the corresponding interval is therefore not greater than $\Delta_\alpha + \Delta_\beta$. Denote this lower limit by L and let L' be some definite positive quantity less than L. It now follows from what has been said that if we choose an arbitrarily small positive σ and any pair of values (y_s, z_t) then there exists a finite number of lines $(y = y_i, z = z_k)$ where the absolute value of $\sqrt{(y_i^2 + z_k^2)}$ is less than the absolute value of $\sqrt{(y_s^2 + z_t^2)}$, and on each such line there exists a finite number of segments each in length greater than L' and whose projections on $(y = y_0, z = z_0)$ completely fill the interval $\alpha \leqq x \leqq \beta$, such that for every point x of these segments

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma.$$

This proves that the condition of the theorem is necessary.

The condition is also sufficient; for choose at pleasure a $\sigma > 0$ and a $y = y_s$ and a $z = z_t$, and let x_0 be any point of the interval (A). Then there exists by hypothesis a finite number of lines $(y = y_i, z = z_k)$, where the absolute value of $\sqrt{(y_i^2 + z_k^2)}$ is less than the absolute value of $\sqrt{(y_s^2 + z_t^2)}$, and on each of these lines there exists a finite number of segments, each in length greater than $L > 0$, such that the projections of these segments on the line $(y = y_0, z = z_0)$ completely fill the interval (A), and such that for every point x of these segments we have

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma.$$

It follows from this that if we construct through the point x_0 a plane parallel to the YZ-plane then this plane must meet at least one of these segments, say a segment on the line $(y = \bar{y}_1, z = \bar{z}_1)$. Now choose $\bar{y}_1$ and $\bar{z}_1$ for our new y_s and z_t respectively. Then by the same reasoning as above there exists for the same σ a finite number of lines $(y = y_i, z = z_k)$ between $(\bar{y}_1, \bar{z}_1)$ and (y_0, z_0) where also the absolute value of $\sqrt{(y_i^2 + z_k^2)}$ is less than the absolute value of $\sqrt{(\bar{y}_1^2 + \bar{z}_1^2)}$, and on each of these lines there exists a finite number of segments each of length greater than L and such that their projections on the line $(y = y_0, z = z_0)$ completely fill the interval (A), such that for every point x of these segments

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma.$$

It follows from this that if we construct through the point x_0 a plane parallel to the YZ-plane then this plane must meet at least one of these segments, say a segment on the line $(y=\bar{y}_2,\ z=\bar{z}_2)$. Now choose $\bar{y}_2$ and $\bar{z}_2$ for a new y_s and z_t respectively and repeat the reasoning. We thus obtain a set of lines $(y=\bar{y}_i,\ z=\bar{z}_k)$ dense at $(y=y_0,\ z=z_0)$, where the absolute value of $\sqrt{(\bar{y}_i^2+\bar{z}_k^2)}$ is less than the absolute value of the same quantity for any preceding $(\bar{y}_i,\ \bar{z}_k)$, and on each of these lines there exists a finite number of segments each of length greater than L, such that the projections of these segments on the line $(y=y_0,\ z=z_0)$ completely fill the interval (A) and such that for every point x of these segments

$$|f(x,\bar{y}_i,\bar{z}_k)-f(x,y_0,z_0)|<\sigma.$$

From this we have at once by theorem 5 of this section that $f(x,y_0,z_0)$ is continuous in x at the point x_0. But x_0 is any point of the interval (A) and hence $f(x,y_0,z_0)$ is continuous in x throughout the interval (A). This proves the theorem completely.

Theorem 7.—If the function $f(x,y,z)$ is continuous in (x,y) together and in z alone throughout the region

$$\alpha \leqq x \leqq \beta,\quad a \leqq y \leqq b,\quad z_0 < z \leqq z_n,$$

and if $\lim\limits_{z=z_0} f(x,y,z)=f(x,y,z_0)$ for every (x,y) of the region

$$\alpha \leqq x \leqq \beta,\quad a \leqq y \leqq b, \tag{A}$$

then $f(x,y,z_0)$ may be discontinuous in (x,y) together at a set of points that is everywhere dense in the region (A).

We shall prove the truth of this theorem by means of an example.

Example 1.—(Cf. Theorem 4 of section 7.)

$$\text{Let } f(x,y,z)=\sum_{n=1}^{\infty}1/n!\left[(\cos n!\,\pi x)^{\frac{2}{z}}+(\cos n!\,\pi y)^{\frac{2}{z}}\right],$$

where

$$0\leqq x<1,\quad 0\leqq y<1,\quad 0<z\leqq 1.$$

We must show first that this example satisfies the conditions of the theorem. It is at once evident that

$$\sum_{n=1}^{\infty}1/n!\left[(\cos n!\,\pi x)^{\frac{2}{z}}+(\cos n!\,\pi y)^{\frac{2}{z}}\right]$$

is continuous in (x,y) together for all values of z different from zero. Let

$z = c \neq 0$. We then have for every value of n and for every point (x, y) of the region

$$0 \leqq x < 1, \quad 0 \leqq y < 1 \tag{A}$$

that
$$\sum_{k=1}^{n} 1/k!\left[(\cos k!\,\pi x)^{\frac{2}{c}} + (\cos k!\,\pi y)^{\frac{2}{c}}\right] \leqq \sum_{k=1}^{n} 2/k!.$$

But $\sum\limits_{k=1}^{\infty} 2/k!$ is a convergent series and hence $\sum\limits_{k=1}^{\infty} 1/k!\left[(\cos k!\,\pi x)^{\frac{2}{c}} + (\cos k!\,\pi y)^{\frac{2}{c}}\right]$ also converges. Furthermore since the series $\sum\limits_{k=1}^{\infty} 2/k!$ converges, it follows from theorem 5 of section 10 that $\sum\limits_{n=1}^{\infty} 1/n!\left[(\cos n!\,\pi x)^{\frac{2}{c}} + (\cos n!\,\pi y)^{\frac{2}{c}}\right]$, considered as a function of x, y, and n, converges uniformly throughout region (A).

Again let $x = x_0$ and $y = y_0$ where x_0 and y_0 are constant values of the given region. Then we have

$$\sum_{k=1}^{n} 1/k!\left[(\cos k!\,\pi x_0)^{\frac{2}{z}} + (\cos k!\,\pi y_0)^{\frac{2}{z}}\right] \leqq \sum_{k=1}^{n} 2/k!.$$

But $\sum\limits_{k=1}^{\infty} 2/k!$ is a convergent series and hence

$$\sum_{n=1}^{\infty} 1/n!\left[(\cos n!\,\pi x_0)^{\frac{2}{z}} + (\cos n!\,\pi y_0)^{\frac{2}{z}}\right]$$

also converges. From theorem 5 of section 10 it also follows that the series just given converges uniformly when considered as a function of n and z. Hence it is continuous in z alone. We have then shown that the series which represents the function is continuous in (x, y) together and in z alone. It still remains to show that $\lim\limits_{z=0} f(x, y, z) = f(x, y, 0)$ for every (x, y) of region (A). Since

$$\sum_{n=1}^{\infty} 1/n!\left[(\cos n!\,\pi x)^{\frac{2}{z}} + (\cos n!\,\pi y)^{\frac{2}{z}}\right],$$

when considered as a function of z, converges uniformly and is therefore continuous in z, it follows that

$$\lim_{z=0} \sum_{n=1}^{\infty} 1/n!\Big[(\cos n!\,\pi x)^{\frac{2}{z}} + (\cos n!\,\pi y)^{\frac{2}{z}}\Big] = \lim_{z=0} f(x, y, z) = f(x, y, 0).$$

It also follows from the uniform convergence of the above function that

$$\lim_{z=0} \sum_{n=1}^{\infty} 1/n!\Big[(\cos n!\,\pi x)^{\frac{2}{z}} + (\cos n!\,\pi y)^{\frac{2}{z}}\Big] = \lim_{z=0}\Big[(\cos \pi x)^{\frac{2}{z}} + (\cos \pi y)^{\frac{2}{z}}\Big]$$

$$+ \lim_{z=0} 1/2!\Big[(\cos 2!\,\pi x)^{\frac{2}{z}} + (\cos 2!\,\pi y)^{\frac{2}{z}}\Big] + \cdots\cdots$$

$$+ \lim_{z=0} 1/k!\Big[(\cos k!\,\pi x)^{\frac{2}{z}} + (\cos k!\,\pi y)^{\frac{2}{z}} + \cdots\cdots$$

Now suppose that x and y are rational, that is to say, that $x = p/q$ where p and q are integers and relatively prime to each other, and that $y = p'/q'$ where p' and q' are integers and relatively prime to each other. If then we begin with the lowest term in the series and pass along it, we shall ultimately reach a term where both q and q' are divisors of $k!$ Consider the terms of the series which precede this term. For each such term $(\cos k!\,\pi x)^2 < 1$ and $(\cos k!\,\pi y)^2 < 1$, hence $\lim\limits_{z=0} (\cos k!\,\pi x)^{\frac{2}{z}} = 0$ and $\lim\limits_{z=0} (\cos k!\,\pi y)^{\frac{2}{z}} = 0$. For the term however for which $k!$ is divisible by q and q' and for all the succeeding terms we have $(\cos k!\,\pi x)^2 + (\cos k!\,\pi y)^2 = 2$. For such values of x and y we therefore have $\lim\limits_{z=0} f(x, y, z) > 0$.

We next suppose that x and y have irrational values. Then no matter how large k is taken we always have $(\cos k!\,\pi x)^2 < 1$ and $(\cos k!\,\pi y)^2 < 1$, and hence $\lim\limits_{z=0} (\cos k!\,\pi x)^{\frac{2}{z}} = 0$ and $\lim\limits_{z=0} (\cos k!\,\pi y)^{\frac{2}{z}} = 0$. For such irrational values of x and y the series therefore always equals zero. We see then that $f(x, y, 0)$ is a function which equals zero for all irrational values of x and y, but differs from zero for all rational values of x and y. It is therefore discontinuous in (x, y) together at all rational points, that is to say at a set of points everywhere dense in the region (A).

In the case of this example it is easy to show that the (x, y) discontinuity cannot be greater than or equal to k (where k is some definite positive number) at every point of the set. From theorem 1 of section 7 it follows that if a function of x and y has an (x, y) discontinuity greater than or equal to k at a set of points that is everywhere dense, then it must have such an (x, y) discontinuity also at the limiting points of the set. The function $f(x, y, 0)$ of the example is a function that has an (x, y) discontinuity at a set of points that is everywhere dense. Hence if this discontinuity were greater than or equal to k at every point of the set, $f(x, y, 0)$ would have an (x, y) discontinuity greater than or equal to k at every point of the region which is impossible.

We now wish to see if the function $f(x, y, z_0)$ can be discontinuous in (x, y) together at every point of the region. It can be proved that this cannot be the case, but before doing so we need a lemma concerning a function of two variables.

Lemma.[*]—If the function $f(x, y)$ is discontinuous in (x, y) together at every point (x, y) of a region, then there always exists a sub-region within which the (x, y) discontinuity is greater than a sufficiently small quantity $A > 0$ at every point.

In order to prove this lemma we must show that the points at which $\lim\limits_{\rho=0} k < \sigma$, where $\sigma > 0$, cannot be everywhere dense in any sub-region, where k represents the oscillation of the function within any circle of radius ρ. Suppose that these points are everywhere dense in some sub-region. Now choose a set of positive quantities

$$\sigma_1 > \sigma_2 > \sigma_3 > \ldots\ldots\ldots > \sigma_n > \ldots$$

where $\lim\limits_{n=\infty} \sigma_n = 0$. Since we have assumed that the points at which $\lim\limits_{\rho=0} k < \sigma$ are everywhere dense in every sub-region, we can within any sub-region find a circle R_0 of radius $r_0 > 0$ about some point (x_0, y_0) such that at every point within this circle the oscillation k of the function is less than σ_1. Hence there exists a circle R_1 of radius r_1 greater than zero and center at (x_1, y_1) and lying wholly within the circle R_0, such that at every point within this circle R_1 the oscillation k is less than σ_2. In a like manner we can determine a circle R_2

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[*] See E. J. Townsend: Ueber den Begriff und die Anwendung des Doppellimes, 1900.

of radius $r_2 > 0$ and center (x_2, y_2) and lying wholly within the circle R_1, such that the oscillation k is less than σ_3 at every point within the circle R_2. Continue this process and we obtain in general a circle R_n of radius r_n greater than zero and center at (x_n, y_n) and lying entirely within the circle R_{n-1}, such that the oscillation k of the function is less than σ_n at every point within the circle R_n. Since now $r_1 > r_2 > r_3 > \cdots\cdots > r_n > \cdots$, we have $\lim\limits_{n=\infty} r_n = 0$, while at the same time the centers of the circles approach a definite limiting point (x_0, y_0). At this point therefore we have $\lim\limits_{\rho=0} k = 0$, where ρ is the radius of any circle about (x_0, y_0). Hence the function $f(x, y)$ is continuous in (x, y) together at this point, which contradicts the hypothesis. The lemma is therefore true.

Theorem 8.*—If the function $f(x, y, z)$ is continuous in (x, y) together and in z alone throughout the region

$$\alpha < x < \beta, \quad a < y < b, \quad z_0 < z < z_n,$$

and if at every point (x, y) of the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

we have $\lim\limits_{z=z_0} f(x, y, z) = f(x, y, z_0)$, then $f(x, y, z_0)$ cannot be discontinuous in (x, y) together at every point of the region

$$\alpha < x < \beta, \quad a < y < b, \tag{B}$$

or of any sub-region thereof.

Suppose that the limiting function $f(x, y, z_0)$ could have an (x, y) discontinuity at every point of region (B). Then there must exist, on $z = z_0$, a circle of radius $r_0 > 0$, such that at every point of this circle the (x, y) discontinuity of the function is greater than a sufficiently small quantity $A > 0$. This follows from the lemma we have just proved. Let (x_0, y_0) be the center of the above circle. Since by hypothesis $\lim\limits_{z=z_0} f(x, y, z) = f(x, y, z_0)$ at every point of the region (A) we can choose an arbitrarily small $\sigma > 0$ and can then find a $z = z_1$, such that

$$| f(x_0, y_0, z_1) - f(x_0, y_0, z_0) | < 1/2\sigma \tag{1}$$

where

$$z_0 < z_1 \leqq z_0 + \theta_0, \quad \theta_0 > 0.$$

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*Baire gives the corresponding theorem for a function of a single variable. See Annali di Matematica, 1899.

But the function $f(x, y, z)$ is by hypothesis continuous in (x, y) together throughout region (B) and for every z for which $f(x, y, z)$ is defined, and hence there must exist for the same σ a circle of radius r_1 about (x_0, y_0) in the plane $z = z_1$, such that for every point within this circle

$$|f(x, y, z_1) - f(x_0, y_0, z_1)| < 1/2\sigma \tag{2}$$

where

$$z_0 < z_1 \leqq z_0 + \theta_0.$$

Let $r_1 < r_0$ so that inequalities (1) and (2) hold for every point (x, y) within this circle and we may therefore add these inequalities and thus obtain

$$|f(x, y, z_1) - f(x_0, y_0, z_0)| < \sigma. \tag{3}$$

This holds for every point (x, y) within the circle of radius r_1 about (x_0, y_0). Since by assumption $f(x, y, z_0)$ is to be discontinuous in (x, y) together at every point of region (B), there must exist in the plane $z = z_0$ within the circle of radius r_1 about (x_0, y_0) at least one point (x_1, y_1), such that

$$|f(x_1, y_1, z_0) - f(x_0, y_0, z_0)| > A. \tag{4}$$

But by hypothesis $\lim\limits_{z = z_0} f(x_1, y_1, z) = f(x_1, y_1, z_0)$ and hence there must exist a plane $z = z_2$, such that for the same σ as above

$$|f(x_1, y_1, z_2) - f(x_1, y_1, z_0)| > 1/2\sigma, \tag{5}$$

where

$$z_0 < z_2 < z_1.$$

Since $f(x, y, z_2)$ is continuous in (x, y) together throughout region (B), there must exist for the same σ in the plane $z = z_2$ a circle of radius r_2 about (x_1, y_1), lying entirely within the circle of radius r_1, such that for every point (x, y) within this circle

$$|f(x, y, z_2) - f(x_1, y_1, z_2)| < 1/2\sigma \tag{6}$$

where

$$z_0 < z_2 < z_1.$$

Let $r_2 < r_1$, so that inequalities (5) and (6) hold for every point (x, y) within the circle of radius r_2. We may therefore add these inequalities and thus obtain

$$|f(x, y, z_2) - f(x_1, y_1, z_0)| < \sigma \tag{7}$$

for every point (x, y) within the circle of radius r_2. We have moreover assumed that $f(x, y, z_0)$ is discontinuous in (x, y) together at every point of region (B) and hence there must exist in the plane $z = z_0$ at least one point (x_2, y_2) within the circle of radius r_2 about (x_1, y_1) for which

$$|f(x_2, y_2, z_0) - f(x_1, y_1, z_0)| > A. \tag{8}$$

We can continue this reasoning indefinitely and have, in general, that there exists a circle of radius r_n about (x_{n-1}, y_{n-1}) and lying wholly within each of the preceding circles and that there also exists a corresponding plane $z = z_n$, where $z_1 > z_2 > z_3 > \dots > z_n > \dots > z_0$, such that for every point (x, y) within this circle the following two inequalities hold

$$|f(x, y, z_n) - f(x_{n-1}, y_{n-1}, z_0)| < \sigma \tag{9}$$

$$|f(x_{n-1}, y_{n-1}, z_0) - f(x_{n-2}, y_{n-2}, z_0)| > A. \tag{10}$$

Now as n increases indefinitely, the radius r_n decreases indefinitely and approaches zero as limit. The centers (x_n, y_n) of the circles must at the same time approach a definite fixed limiting point $(\bar{x}, \bar{y})$. We thus obtain the following inequalities

$$|f(x_0, y_0, z_0) - f(\bar{x}, \bar{y}, z_1)| < \sigma \qquad \text{(from 3)}$$

$$|f(\bar{x}, \bar{y}, z_2) - f(x_1, y_1, z_0)| < \sigma \qquad \text{(from 7)}$$

$$|f(x_1, y_1, z_0) - f(x_0, y_0, z_0)| > A.$$

If now we choose $0 < \sigma < 1/4\,A$ and combine these three inequalities we obtain

$$|f(\bar{x}, \bar{y}, z_2) - f(\bar{x}, \bar{y}, z_1)| > 1/2\,A.$$

In a similar manner we obtain the following inequalities

$$|f(\bar{x}, \bar{y}, z_3) - f(\bar{x}, \bar{y}, z_2)| > 1/2\,A$$

$$|f(\bar{x}, \bar{y}, z_4) - f(\bar{x}, \bar{y}, z_3)| > 1/2\,A$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$|f(\bar{x}, \bar{y}, z_n) - f(\bar{x}, \bar{y}, z_{n-1})| > 1/2\,A.$$

But the z's are dense at $z = z_0$ and hence we obtain from the last inequality

$$\lim_{\substack{z_{n-1} = z_0 \\ z_n = z_0}} [f(\bar{x}, \bar{y}, z_n) - f(\bar{x}, \bar{y}, z_{n-1})] > 1/2\,A \neq 0.$$

and hence the limit $\lim\limits_{z = z_0} f(x, y, z)$ does not exist at the point $(\bar{x}, \bar{y})$. The assumption that $f(x, y, z_0)$ has an (x, y) discontinuity at every point of region (B) is therefore false and the theorem is true.

Remark 1.—In the above theorem we assumed that $\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$ for every point of region (A), but we were only able to prove that the limiting function $f(x, y, z_0)$ is not discontinuous in (x, y) together at every point of region (B). As a matter of fact, $f(x, y, z_0)$ may have an (x, y) discontinuity at every point of the boundaries of region (A). We show this by means of the following example.

Example 2. Let $f(x, y, z) = \dfrac{(1 + \sin \pi xy)^{1/z} - 1}{(1 + \sin \pi xy)^{1/z} + 1}$,

where $0 \leqq x \leqq 10$, $0 \leqq y \leqq 10$, $0 < z \leqq 10$, and $\lim\limits_{z=0} f(x, y, z) = f(x, y, 0)$.

The value of the limit $\lim\limits_{z=0} f(x, y, z)$ is evidently zero for all values of x and y which make xy integral, for in this case $\sin \pi xy = 0$. But if we let $2n < xy < 2n + 1$, where n is some integer, then for all such points we have $0 < \sin \pi xy < 1$. Hence for all such values of x and y

$$\lim_{z=0} \frac{(1 + \sin \pi xy)^{1/z} - 1}{(1 + \sin \pi xy)^{1/z} + 1} = \lim_{z=0} \frac{1 - \dfrac{1}{(1 + \sin \pi xy)^{1/z}}}{1 + \dfrac{1}{(1 + \sin \pi xy)^{1/z}}} = +1.$$

On the other hand if we let $2n + 1 < xy < 2n + 2$, where n is a positive integer, then we obtain

$$\lim_{z=0} \frac{(1 + \sin \pi xy)^{1/z} - 1}{(1 + \sin \pi xy)^{1/z} + 1} = \frac{0 - 1}{0 + 1} = -1.$$

But when either x or y is equal to zero, then $f(x, y, 0) = 0$ along the whole X-axis and the whole Y-axis. Hence $f(x, y, 0)$ is discontinuous in (x, y) together at every point of these lines.

Remark 2—In theorem 7 we showed that, for the example there given, the (x, y) discontinuity could not be greater than or equal to k at every point. We are now in a position to prove in general that, if the function $f(x, y, z)$ satisfies the conditions of theorem 7, then the limiting function $f(x, y, z_0)$ cannot have an (x, y) discontinuity greater than or equal to k at a set of points that is everywhere dense. For if this were possible, then it follows from theorem 1 of section 7 that the function $f(x, y, z_0)$ has an (x, y) discontinuity greater than or equal to k also at the limiting points, that is to say it would have such an (x, y) discontinuity at every point of the region. This is however impossible by theorem 8 just proved.

Theorem 9.—If the function $f(x, y, z)$ is continuous in x alone and in (y, z) together throughout the region

$$\alpha < x < \beta, \quad y_0 < y < y_m, \quad z_0 < z < z_n,$$

and if for every point of the interval $\alpha \leqq x \leqq \beta$, $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$, then $f(x, y_0, z_0)$ cannot be discontinuous in x alone at every point of the interval $\alpha < x < \beta$.

The proof of this theorem is strictly analogous to that of theorem 8. We replace the circles by intervals about x_0 on lines parallel to the X-axis, in a manner similar to that employed in theorem 6.

Theorem 10.—If the function $f(x, y, z)$ is continuous in (x, y) together and in z alone throughout the region

$$\alpha < x < \beta, \; a < y < b, \; z_0 < z < z_n,$$

and if at every point (x, y) of the region

$$\alpha \leqq x \leqq \beta, \; a \leqq y \leqq b, \tag{A}$$

$\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$, then the function $f(x, y, z_0)$ must be continuous in (x, y) together at a set of points that is at least dense everywhere in the region

$$\alpha < x < \beta, \; a < y < b. \tag{B}$$

Suppose that this is not so. Then there must exist in the plane $z = z_0$ some circle R_0 of radius r_0 greater than zero and lying entirely within region (B), such that within this circle there is no point for which $\lim\limits_{\rho = 0} k = 0$, where ρ is the radius of any variable circle about the point and k is the oscillation of the function within this circle. Within the circle R_0 therefore $\lim\limits_{\rho = 0} k \neq 0$ for every point. Hence every point (x, y) within the circle R_0 is a point at which $f(x, y, z_0)$ has an (x, y) discontinuity. This however contradicts theorem 8, and hence the theorem is true.

This theorem may be extended in a manner similar to that employed in the extension of theorem 5 of section 7. We thus obtain the following corollary.

Corollary.—If the function $f(x, y, z)$ satisfies the conditions of theorem 10, then the points at which $f(x, y, z_0)$ is continuous in (x, y) together form a set of the second category, and hence a non-enumerable set.

Theorem 11.—If the function $f(x, y, z)$ is continuous in x alone and in (y, z) together throughout the region

$$\alpha < x < \beta, \quad y_0 < y < y_m, \quad z_0 < z < z_n,$$

and if $\lim\limits_{\substack{y=y_0 \\ z=z_0}} f(x, y, z) = f(x, y_0, z_0)$ for every point of the interval $\alpha \leqq x \leqq \beta$, then $f(x, y_0, z_0)$ is continuous in x alone at a set of points that is at least everywhere dense in the interval $\alpha < x < \beta$. Moreover this set belongs to the second category and is non-enumerable.

Suppose that this is not true. Then there must exist on the X-axis some interval greater than zero and lying wholly within the interval $\alpha < x < \beta$, such that within this interval there is no point x for which the limit of the oscillation of the function $f(x, y_0, z_0)$ is zero. Within this interval therefore the function $f(x, y_0, z_0)$ has an x discontinuity at every point. But this contradicts theorem 9, and hence our theorem is true as regards the first part.

The second part of the theorem is proved in exactly the same manner as was the corollary of the preceding theorem.

CHAPTER III.

SERIES OF FUNCTIONS OF TWO REAL VARIABLES.

10. *Properties of Series of Functions of Two Real Variables.*—Suppose that we have given

$$S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \dots + u_n(x, y),$$

where each u is finite and a continuous function of (x, y) together throughout a certain region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b,$$

for every value of n $(n = 1, 2, 3, \dots)$. Since each u is finite and continuous in (x, y) together and since n is also finite, it follows that $S_n(x, y)$ must be finite and continuous in (x, y) together throughout the given region. If then for every constant value of x and y of the region, $\lim_{n=\infty} S_n(x, y) = S(x, y)$, then $S(x, y)$ represents a function of (x, y) throughout the given region, and the question now arises as to the properties of this limiting function $S(x, y)$.

If we wish, we may consider $S_n(x, y)$ also as a function of the three variables x, y, and n. This enables us to apply the results of the preceding chapters to the present problem. In order to do so, we define x and y as above, namely as any values of

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b.$$

In order to define z we put $z = 1/n$, so that the z of the preceding chapters takes on the values $z = 1, 1/2, 1/3, \ldots, 1/n \ldots$, which is a set of values dense at $z = 0$. We then have

$$S(x, y) = \lim_{n=\infty} S_n(x, y) = \lim_{z=0} f(x, y, z),$$

where $f(x, y, z) = S_n(x, y)$ for every point (x, y) of the given region. It follows from this that everything which was shown to be true for the limiting function $f(x, y, z_0)$ in the case of a function of three variables must also hold for the series as defined above. The converse of this is, however, not true, for if we have proved a theorem about an infinite series, it does not follow necessarily that an analogous theorem exists for a function of three variables. The reason for this is to be found in the fact that, in the case of a function of three variables, z is defined for all positive values, while in the case of the series it is defined only for a set of values dense at $z = 0$. It may therefore happen that a theorem holds for a set of values dense at $z = z_0$, where $z_0 < z \leqq z_0 + \theta_0$, while the theorem will not hold for *all* values z of the interval $z_0 < z \leqq z_0 + \theta_0$.

From what has been said follows that to the three different kinds of uniform convergence of $f(x, y, z)$ to its limiting function, as they were defined in the last chapter, there correspond three different kinds of uniform convergence of $S_n(x, y)$ to $S(x, y)$. These are as follows:

(1) If $S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \ldots + u_n(x, y)$ is, for every finite value of n, finite and continuous in (x, y) together throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

and if for every point (x, y) of this region $\lim_{n=\infty} S_n(x, y) = S(x, y)$, then $S_n(x, y)$

converges uniformly to $S(x, y)$ throughout region (A), if having chosen an arbitrarily small $\sigma > 0$, we can then find a positive integer m, such that

$$|R_n(x, y)| \equiv |S_n(x, y) - S(x, y)| < \sigma,$$

when $n > m$ and for all points (x, y) of region (A).

(2) If $S_n(x, y)$ and $S(x, y)$ are defined as in (1), then $S_n(x, y)$ converges "einfach gleichmässig" throughout region (A) to $S(x, y)$, if having chosen an arbitrarily small $\sigma > 0$ we can then find a positive integer m, such that

$$|R_n(x, y)| \equiv |S_n(x, y) - S(x, y)| < \sigma,$$

for some set of integral values n greater than m, which however increase without limit. In this case the n's do not assume all integral values.

(3) If $S_n(x, y)$ and $S(x, y)$ are defined as in (1), then $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout the region (A), if having chosen an arbitrarily small $\sigma > 0$, there then exists a plane $z = 1/n$ and between this plane $z = 1/n$ and the plane $z = 0$ a finite number of planes, and on each of these a finite number of rectangles, having their sides parallel to the coordinate axes, such that their projections on the plane $z = 0$ completely fill the region (A) and that for every point (x, y) of these rectangles

$$|R_n(x, y)| \equiv |S_n(x, y) - S(x, y)| < \sigma.$$

We will now give the theorems that deal with the properties of a series of functions of two real variables. Unless something is said to the contrary, we shall in every case make the following assumptions about the series:

(1) $$S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \dots + u_n(x, y),$$

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \qquad \text{(A)}$$

where $u_i(x, y)$, $(i = 1, 2, 3, \dots, n)$ is continuous in (x, y) together throughout region (A);

(2) $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$, for every constant pair of values (x, y) of region (A).

From theorem 1 of section 8 we have immediately

Theorem 1.—The necessary and sufficient condition that $S_n(x, y)$ shall converge uniformly to $S(x, y)$ throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \qquad \text{(A)}$$

is that at every point (x_0, y_0) of this region, boundaries included, the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ n=\infty}} S_n(x, y)$ exists and equals $S(x_0, y_0)$.

Example 1.

$$\text{Let } S_n(x, y)=\frac{xy}{xy+1}+\left[\frac{2xy}{4xy+1}-\frac{xy}{xy+1}\right]+\cdots\cdot$$
$$+\left[\frac{nxy}{n^2xy+1}-\frac{(n-1)xy}{(n-1)^2xy+1}\right]=\frac{nxy}{n^2xy+1}$$

where $0\leqq x\leqq 1,\quad 0\leqq y\leqq 1.$

In this case the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ n=\infty}} S_n(x, y)$ exists and equals zero at every point (x, y) of the region, and hence $S_n(x, y)$ converges uniformly to $S(x, y)$ throughout the region.

Example 2.

$$\text{Let } S_n(x, y)=\frac{xy}{x^3+y^3+1}+\left[\frac{4xy}{8x^3+8y^3+1}-\frac{xy}{x^3+y^3+1}\right]+\cdots\cdot$$
$$+\left[\frac{n^2xy}{n^3x^3+n^3y^3+1}-\frac{(n-1)^2xy}{(n-1)^3x^3+(n-1)^3y^3+1}\right]$$
$$=\frac{n^2xy}{n^3x^3+n^3y^3+1}$$

where $0\leqq x\leqq 1,\quad 0\leqq y\leqq 1.$

Here the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ n=\infty}} S_n(x, y)$ does not exist for $x=y=0$ and hence $S_n(x, y)$ does not converge uniformly to $S(x, y)$.

From theorem 1 of section 9 follows:

Theorem 2.—If $S_n(x, y)$ converges uniformly to $S(x, y)$ throughout the region

$$\alpha\leqq x\leqq\beta,\quad a\leqq y\leqq b,$$

then $S(x, y)$ is continuous in (x, y) together throughout the given region.

This condition is only a sufficient condition for the continuity of $S(x, y)$ in (x, y) together. Thus in example 2 the function $S(x, y)$ is continuous in (x, y) together throughout the region given, but $S_n(x, y)$ does not converge uniformly to $S(x, y)$.

The necessary and sufficient condition for the continuity of $S(x, y)$ in (x, y) together at some fixed point (x_0, y_0) of the region follows from theorem 3 of section 9. It may be stated as follows:

Theorem 3.—The necessary and sufficient condition that the function $S(x, y)$ is continuous in (x, y) together at a fixed point (x_0, y_0) of the given region, is that having chosen an arbitrarily small $\sigma > 0$ and having some set of positive integers $m_1, m_2, m_3, \ldots, m_k, \ldots$, which increase indefinitely, there shall then exist about (x_0, y_0) corresponding to each m_k a circle of radius $r_k > 0$ (which may vary with m_k), such that for all points (x, y) within this circle

$$|S_{m_k}(x, y) - S(x, y)| < \sigma.$$

From theorem 5 of section 9 we obtain the necessary and sufficient condition that $S(x, y)$ is continuous in (x, y) together throughout the given region. The condition may be stated as follows:

Theorem 4.—The necessary and sufficient condition that $S(x, y)$ is continuous in (x, y) together at every point (x, y) of the region

$$\alpha \leq x \leq \beta, \quad a \leq y \leq b. \qquad \text{(A)}$$

is that for every arbitrarily small $\sigma > 0$ there shall exist a plane $z = 1/n$ and between this plane and the plane $z = 0$ a finite set of planes parallel to the XY-plane, each of these to contain a finite set of rectangles whose sides are parallel to the coordinate axes and whose projections on the plane $z = 0$ completely fill the region (A), such that for every point (x, y) of these rectangles

$$|S_n(x, y) - S(x, y)| < \sigma.$$

Theorem 5.—If the sum of the maximum values of the terms of a series $S_n(x, y)$ converges absolutely within a given region, then the series converges also uniformly.

The following conditions are given:

(1) $S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \ldots + u_n(x, y)$

$$\alpha \leq x \leq \beta, \quad a \leq y \leq b. \qquad \text{(A)}$$

Here as elsewhere we confine ourselves to the case where $u_i(x, y)$ is continuous in (x, y) together;

(2) $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$ for every pair of values (x, y) of region (A);

(3) $|u_i(x, y)| = M_i$, where M_i is positive and finite, and this is to hold for every (x, y) of region (A);

(4) $\sum\limits_{i=1}^{\infty} M_i$ converges.

We are to show that under these conditions $S_n(x, y)$ converges uniformly throughout region (A).

Since $\sum\limits_{i=1}^{\infty} M_i$ is convergent, we can choose an arbitrarily small $\varepsilon > 0$ and can then find an m, such that

$$(M_{m+1} + M_{m+2} + \ldots + M_{m+p}) < \varepsilon, \quad p = 1, 2, 3, \ldots\ldots$$

Hence for m sufficiently large

$$|u_{m+1}(x, y) + u_{m+2}(x, y) + \ldots + u_{m+p}(x, y)| < \varepsilon, \quad p = 1, 2, \ldots$$

for every point (x, y) of region (A). We therefore have at every point of this region

$$|G - S_n(x, y)| < \varepsilon, \text{ where } n > m,$$

and hence the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ n=\infty}} S_n(x, y)$ exists at every point of region (A), boundaries included.

From 4 of the hypothesis follows that $S_n(x, y)$ also converges and hence by virtue of theorem 3 of section 2

$$\lim_{\substack{x=x_0\\ y=y_0\\ n=\infty}} S_n(x, y) = \lim_{n=\infty} \lim_{\substack{x=x_0\\ y=y_0}} S_n(x, y).$$

But $S_n(x, y)$ is continuous in (x, y) together and hence

$$\lim_{n=\infty} \lim_{\substack{x=x_0\\ y=y_0}} S_n(x, y) = \lim_{n=\infty} S_n(x, y) = S(x_0, y_0)$$

at every point (x_0, y_0) of region (A). The simultaneous limit therefore not only exists, but it also equals the value of the function at the point, and hence the series converges uniformly by theorem 1 of this section.

In theorem 3 we have given the necessary and sufficient condition that the limiting function $S(x, y)$ is continuous in (x, y) together. The question now presents itself as to whether the limiting function may be discontinuous and, if it is discontinuous, how often it may be discontinuous. We obtain an answer to this question from theorem 7 of section 9. It is as follows:

Theorem 6.—If $S_n(x, y)$ and $S(x, y)$ are defined as in theorem 1, then the limiting function $S(x, y)$ may be discontinuous in (x, y) together at a set of points that is everywhere dense in the given region, but in this case the (x, y) discontinuities cannot all be greater than or equal to some finite positive number $\bar{k}$.

The proof follows immediately from the example under theorem 7 of section 9. If in this we let $z = 1/n$, then the example becomes

$$S_n(x, y) = \sum_{k=1}^{n} 1/k!\left[(\cos k!\,\pi x)^{2n} + (\cos k!\,\pi y)^{2n}\right],$$

$$0 \leqq x < 1, \quad 0 \leqq y < 1.$$

From theorem 8 of section 9 we obtain

Theorem 7.—If $S_n(x, y)$ and $S(x, y)$ are defined as before, then the limiting function $S(x, y)$ cannot be discontinuous in (x, y) together at every point of the region $\alpha < x < \beta$, $a < y < b$.

From theorem 10 of section 9 we have

Theorem 8.—If $S_n(x, y)$ and $S(x, y)$ are defined as before, then $S(x, y)$ must be continuous in (x, y) together at a set of points that is at least everywhere dense in the region $\alpha < x < \beta$, $a < y < b$. Moreover this set of points is a set of the second category and hence is non-enumerable.

11. *The Integration of Series Term by Term.*

In this section we shall consider the following question. Suppose we know:

(1) that $S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \ldots + u_n(x, y)$, and that the simultaneous integral $\int_\alpha^\beta\int_a^b u_k(x, y)\,dxdy$ exists for every finite value of k. Then the simultaneous integral $\int_\alpha^\beta\int_a^b S_k(x, y)\,dxdy$ also exists for every finite value of k, where our region is defined as

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b. \tag{A}$$

(2) We have also $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$ for every constant pair of values (x, y) of region (A).

This being the case the question arises under what conditions will the integral of $S(x, y)$ exist over the region, and if it exists, under what conditions will it equal the term by term integral? This is to say, under what conditions do we have

$$\int_\alpha^\beta \int_a^b \lim_{n=\infty} S_n(x, y)\, dxdy = \lim_{n=\infty} \int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy?$$

Before answering these questions, let us first explain what we mean by the simultaneous integral. In dealing with a function of two variables we meet with two kinds of integrals, the twice-taken or double integral and the simultaneous integral.

The twice-taken or double integral is defined as the result of integrating the function $f(x, y)$ first with respect to one of the variables, say x, between the given limits (α, β), and then integrating the resulting function with respect to y between the limits (a, b). This gives us the integral which is denoted by $\int_a^b dy \int_\alpha^\beta f(x, y)\, dx$. The integral in question is equivalent to a double limit.

The simultaneous integral $\int_\alpha^\beta \int_a^b f(x, y)\, dxdy$ may be defined as follows: Divide the region into a set of rectangles by means of lines parallel to the X and Y axes. Then multiply the area of each rectangle by the value of the function at some point of the corresponding rectangle, and add the results. We then have

$$\Sigma = (x_0 - \alpha)(y_0 - a) f(\bar{x}_0, \bar{y}_0) + (x_1 - x_0)(y_0 - a) f(\bar{x}_1, \bar{y}_0) + \ldots\ldots\ldots$$
$$+ (x_i - x_{i-1})(y_k - y_{k-1}) f(\bar{x}_i, \bar{y}_k) + \ldots\ldots + (\beta - x_{l-1})(b - y_{m-1}) f(\bar{x}_l, \bar{y}_m)$$

where $f(\bar{x}_i, \bar{y}_k)$ represents the value of the function at some point of the corresponding rectangle. Now take the limit of this sum as l and m increase indefinitely, while at the same time the dimensions of the rectangles decrease indefinitely. We then define the simultaneous integral as the limit thus obtained. The simultaneous integral is therefore equivalent to a simultaneous limit.

In the following pages we shall consider simultaneous integrals as thus defined. We must now answer the following question. If the simultaneous integral $\int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy$ exists, what is the necessary and sufficient condition that the simultaneous integral $\int_{\alpha}^{\beta}\int_{a}^{b} S(x, y)\, dxdy$ shall also exist? This condition is given in the following theorem.

Theorem 1.*—If (1) $S_n(x, y) = \sum_{k=1}^{n} u_k(x, y)$, for all values x and y of the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b; \tag{A}$$

(2) $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$ for every constant pair of values (x, y) of region (A), where both $S_n(x, y)$ and $S(x, y)$ are finite;

(3) the simultaneous integral $\int_{\alpha}^{\beta}\int_{a}^{b} S_n(x, y)\, dxdy$ exists for every constant value of n;

then the necessary and sufficient condition that the simultaneous integral $\int_{\alpha}^{\beta}\int_{a}^{b} S(x, y)\, dxdy$ shall exist is that, having chosen an arbitrarily small $\varepsilon > 0$ and $\sigma > 0$, independent of each other, and having also chosen an $n = m_1$ we can then always find an $m_2 > m_1$, such that for all values (x, y) of region (A), excepting at most for those contained within a set of rectangles finite in number and of total area less than ε, and for some number m which may vary with (x, y) but where always $m_1 < m < m_2$, we have

$$|R_m(x, y)| \equiv |S(x, y) - S_m(x, y)| < \sigma.$$

This condition may be expressed otherwise as follows: Choose as coordinate axes the axes X, Y, and $1/n$ and choose ε and σ as above. We must then be able to find a plane $1/n = m_s$ and between this plane and the plane $1/n = 0$ a finite number of planes, $1/n = m_1, m_2, m_3, \ldots, m_p$, such that after having cut out from these within the given region a finite number of rectangles of total area less than ε, these rectangles to have their sides parallel to the coordinate axes X and Y, the function $S_n(x, y)$ shall converge uniformly by areas to $S(x, y)$ throughout the rest of the region (A). It follows at once from theorem 4 of section 1 that $S(x, y)$ must be continuous in (x, y) together throughout the rest of the region. If the above conditions are fulfilled, then we say with Arzela that $S_n(x, y)$ "converges uniformly by areas in general" to $S(x, y)$.

The condition of the theorem is necessary. We suppose then that the simultaneous integral $\int_{\alpha}^{\beta}\int_{a}^{b} S(x, y)\,dxdy$ exists over the region. The set of points at which $S(x, y)$ has an (x, y) discontinuity greater than or equal to σ, can therefore be at most a discrete set, that is to say, these points must be such that they can be enclosed in a set of rectangles finite in number and of total area less than ε. Let us denote this set by $t_1^{(0)}, t_2^{(0)}, t_3^{(0)}, \ldots, t_p^{(0)}$. Since now the simultaneous integral $\int_{\alpha}^{\beta}\int_{a}^{b} S_n(x, y)\,dxdy$ also exists on each of the planes $1/n = m_1, m_2, \ldots$ the points at which $S_{m_k}(x, y)$, $(k = 1, 2, 3, \ldots)$, has an (xy) discontinuity greater than or equal to σ can form at most a discrete set, and hence

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*See Arzela, "Sulle serie di funzioni". R. Accademia dell 'Instituto di Bologna, 1900.

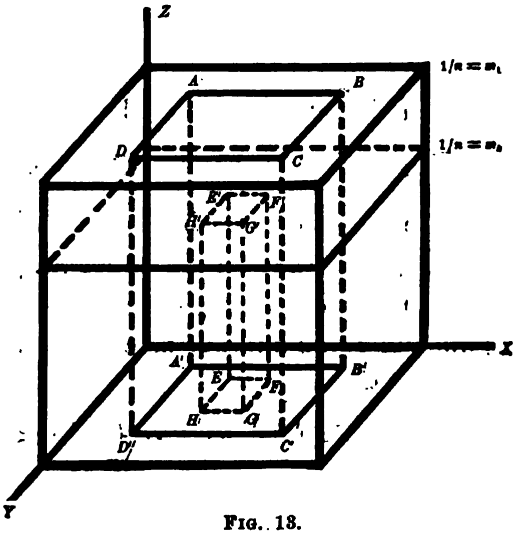

FIG. 13.

on each of the planes $1/n = m_1, m_2, m_3, \ldots$, this set of points can be enclosed in a finite set of rectangles of total area less than ε. Denote this set of rectangles by $t_1^{(k)}, t_2^{(k)}, t_3^{(k)}, \ldots, t_q^{(k)}$, for any plane $1/n = m_{m_k}$. Now let $ABCD$ be one of the rectangles $t^{(1)}$ in the plane $1/n = m_1$ and let its projection on the plane $1/n = 0$ be $A'B'C'D'$. See figure 13.

We can now assert that if $A'B'C'D'$ does not contain any of the rectangles $t^{(0)}$ or a part of such a rectangle, then as we pass from $1/n = m_1$ to $1/n = m_2$, to $1/n = m_3$, etc., we shall ultimately reach a plane $1/n = m_k$, such that the projections of $ABCD$ on $1/n = m_k$ and on all planes between this and the plane $1/n = 0$, do not contain a rectangle $t^{(k)}$ or a part of such a rectangle. In other words, we can find a number $1/n = m_k$ such that for this and all values between this and $1/n=0$ the function $S_k(x, y)$ is continuous in (x, y) together throughout the corresponding rectangle. Suppose namely that this is not the case. Then as we pass through the values $1/n=m_1, m_2, m_3, \ldots,$ we always have a rectangle in every plane or at least in a set of planes dense at $1/n=0$, such that this rectangle lies within the projection of $ABCD$ on the plane in question and furthermore such that within this rectangle the function $S_n(x, y)$ has (x, y) discontinuities greater than or equal to σ. Let us denote the set of rectangles we thus obtain by $T_1, T_2, T_3, \ldots, T_k, \ldots$ Two cases may now arise: either T_k has a lower limit T greater than zero or it has the lower limit zero as k increases indefinitely.

In the first case let $T=EFGH$ (see figure 13). Then no matter how large we take k, we always have in the plane $1/n=m_k$ a rectangle $E'F'G'H'$, corresponding to $EFGH$, such that within this rectangle the function $S_n(x, y)$ has (x, y) discontinuities greater than or equal to σ. Let (x_0, y_0) be a point of this rectangle at which the (x, y) discontinuity is greater than or equal to σ. Then we have for all values of n and for some point (x, y) of this rectangle

$$|S_n(x, y)-S_n(x_0, y_0)|>\sigma. \tag{1}$$

Since by hypothesis $\lim\limits_{n=\infty} S_n(x, y)=S(x, y)$, we have also

$$|S(x, y)-S_n(x, y)|<1/4\sigma; \quad n>m_k. \tag{2}$$

Furthermore since $S(x, y)$ is continuous in (x, y) together throughout the rectangle $EFGH$, we have

$$|S(x_0, y_0)-S(x, y)|<1/4\,\sigma. \tag{3}$$

Combining these three inequalities we obtain

$$|S_n(x_0, y_0)-S(x_0, y_0)|>1/4\,\sigma; \quad n>m_k. \tag{4}$$

Inequality 4 however contradicts the hypothesis that $\lim\limits_{n=\infty} S_n(x, y)=S(x, y)$ for every pair of values (x, y) of region (A), and hence the assumption we made is false. It therefore follows that, if we have in each of a set of planes dense at $1/n=0$ a rectangle of lower limit $T\neq 0$, within which the function $S_n(x, y)$ has an (x, y) discontinuity greater than or equal to σ, then the function $S(x, y)$ has such an (x, y) discontinuity within the rectangle T in the plane $1/n=0$.

Suppose in the second place that the limit of T is zero. We have then in the plane $1/n=m_t$ a rectangle T_t within which any (x, y) discontinuity of

$S_n(x, y)$ is greater than or equal to σ. Furthermore as s increases indefinitely the area of this rectangle approaches zero. Let this occur at the point (x', y') and let (x'', y'') be any other point of the rectangle T_s. If now we drop a perpendicular from (x'', y'') to the plane $1/n = 0$, then one of two things will happen; either there exists a plane $1/n = m_k$, such that the perpendicular does not meet a rectangle $t^{(i)}$ in any plane $1/n = m_i$ between $1/n = m_k$ and $1/n = 0$, or no matter what plane $1/n = m_k$ we choose, the perpendicular will always meet a rectangle $t^{(k)}$. We must now show that in the latter case the perpendicular also meets one of the rectangles $t^{(0)}$ in the plane $1/n = 0$. We assume that the perpendicular meets a rectangle $t^{(k)}$ in every plane $1/n = m_k$. Then for every point of this rectangle

$$|S_{m_k}(x, y) - S_{m_k}(x'', y'')| > \sigma. \tag{1}$$

Since moreover $\lim_{n = \infty} S_n(x, y) = S(x, y)$ for every point of region (A), it follows that

$$|S(x, y) - S_{m_k}(x, y)| < 1/4\,\sigma \tag{2}$$

for every m_k greater than some m'. Likewise

$$|S_{m_k}(x'', y'') - S(x'', y'')| < 1/4\,\sigma \tag{3}$$

for every m_k greater than some m''. Let $m'' > m'$; then for every m_k greater than m'', (1), (2), and (3), may be added and hence

$$|S(x, y) - S(x'', y'')| > 1/4\,\sigma \tag{4}$$

for every (x, y) of some rectangle about (x'', y'') in the plane $1/n = 0$. Hence $S(x, y)$ is discontinuous in (x, y) together at (x'', y'') and the perpendicular from (x'', y'') therefore must meet one of the rectangles $t^{(0)}$ in the plane $1/n = 0$. In a like manner we can prove that the perpendicular from (x', y') meets one of the rectangles $t^{(0)}$ in the plane $1/n = 0$.

We are now ready to complete the proof of the original theorem. In order to do so, let us cut out from the plane $1/n = 0$ all the rectangles $t^{(0)}$, that contain the points at which $S(x, y)$ has an (x, y) discontinuity greater than or equal to σ. As has been shown, this set of rectangles is finite in number and of total area less than ε. Lines through the vertices of these rectangles parallel to the X and Y axes divide the remaining part of region (A) into a finite number of rectangles. Number these 1, 2, 3, ..., q. Now consider rectangle number 1. From what has been proved it follows that there must exist a plane $1/n = m_k$ such that for this plane and for all planes between this and the plane $1/n = 0$

the function $S_n(x, y)$ is continuous in (x, y) together throughout the rectangle in question. Hence by theorem 4 of section 10 the function $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout this rectangle. In a like manner we prove that the function $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout each of the remaining rectangles $2, 3, \ldots, q$. We have thus shown that, after we have cut out from region (A) a finite number of rectangles of total area less than ε, the function $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout the rest of region (A). This proves that the condition of the theorem is necessary.

The condition is also sufficient. By hypothesis then $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout region (A), excepting at the points of a finite number of rectangles of total area less than ε. We can therefore choose at pleasure a plane $1/n = m_s$ and there will then exist between this plane and the plane $1/n = 0$ a finite number of planes having the following properties. From these planes, including the plane $1/n = 0$, are cut out a finite number of rectangles, $t_1, t_2, t_3, \ldots, t_p$, of total area less than ε. These rectangles must be such that, from some m_s on, their projections on the plane $1/n = 0$ cannot completely fill region (A), otherwise the function $S(x, y)$ would be totally discontinuous in (x, y) together throughout region (A). In the remaining part of each plane there exists a finite number of adjacent rectangles (each of area greater than $G > 0$) whose projections on the plane $1/n = 0$ completely fill the remaining part of region (A), such that for every point (x, y) of these rectangles

$$|S_k(x, y) - S(x, y)| < \sigma. \tag{1}$$

Let us denote $S_k(x, y)$ for the points of these rectangles by $K(x, y)$. Then the following inequality always is true

$$|K(x, y) - S(x, y)| < \sigma. \tag{2}$$

Moreover the simultaneous integral $\int_\alpha^\beta \int_a^b K(x, y)\, dxdy$ exists throughout the region for which $K(x, y)$ is defined. Now consider the identity

$$S(x, y) \equiv K(x, y) + S(x, y) - K(x, y),$$

which holds for all points (x, y) for which $K(x, y)$ is defined. In the right member of this identity $K(x, y)$ may have an (x, y) discontinuity greater than or equal to σ only at a discrete set of points. The function $[S(x, y) - K(x, y)]$ can have such an (x, y) discontinuity only at points of the rectangles

$$t_1, t_2, t_3, \ldots, t_p.$$

It follows from this that the sum $K(x, y) + S(x, y) - K(x, y)$ can have such an (x, y) discontinuity only at a discrete set of points. This being the case, it is at once evident that the simultaneous integral $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ exists.

The theorem just proved holds also in the general case of a function of three variables, the proof being similar to that just given.

Having thus obtained the necessary and sufficient condition that the simultaneous integral $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ exists, the next problem is the following.

If the simultaneous integral

$$\int_\alpha^\beta \int_a^b \lim_{n=\infty} S_n(x, y)\, dxdy = \int_\alpha^\beta \int_a^b S(x, y)\, dxdy = \theta(x, y)$$

exists for every point (x, y) of region (A) and if it is finite and continuous, under what conditions will the simultaneous integral $\lim\limits_{n=\infty} \int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy$ exist and equal $\theta(x, y)$?

We have here two cases to consider. For every fixed value of n but for a variable pair of values (x, y) the function $|S_n(x, y)|$ has an upper limit $G > 0$. This G is a function of n, if we consider n as variable. Then we have the following two cases:

(1) G has an upper finite limit G' as n increases indefinitely;

(2) G has no finite upper limit as n increases indefinitely.

These two cases give rise to different conditions which are stated in the following theorems. Arzela* has proved the analogous theorems for a series of functions of a single variable. The proofs given below are generalizations of Arzela's proofs.

Theorem 2.—If $S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \dots + u_n(x, y)$ where each of the simultaneous integrals $\int_\alpha^\beta \int_a^b u_k(x, y)\, dxdy$ exists throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

and if also $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$ for every (x, y) of region (A), and if for every constant value of n and for every (x, y) of region (A) we have $S_n(x, y) < G$, where G is finite, then the necessary and sufficient condition that the simultaneous integral $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ shall exist at every point of region (A) and that at the same time

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* Cf. Arzela: "Sulle serie di funzioni"; R. Accademia dell' Instituto di Bologna, 1900.

$$\int_\alpha^\beta \int_a^b S(x, y)\, dxdy = \lim_{n=\infty} \int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy$$

is that $S(x, y)$ shall converge uniformly by areas in general to $S(x, y)$ throughout region (A).

That the condition is necessary follows immediately from theorem 1 of this section.

The condition is also sufficient. By hypothesis therefore $S_n(x, y) < G$ for every constant n and for every (x, y) of region (A). Hence, since $\lim_{n=\infty} S_n(x, y) = S(x, y)$

$$|S(x, y) - S_n(x, y)| < 2G \tag{1}$$

so that $S(x, y)$ also has a finite upper limit. Let us now choose independently of each other an arbitrarily small $\sigma < 0$ and $\varepsilon < 0$. Since $S_n(x, y)$ converges uniformly by areas in general to $S(x, y)$ throughout region (A), we can find an s such that the points at which

$$|S(x, y) - S_{s+p}(x, y)| \geqq \sigma, \quad (p = 1, 2, 3, \ldots) \tag{2}$$

form a discrete set, that is they can be enclosed in a finite number of rectangles of total area less than ε. Consider now the function $[S(x, y) - S_{s+p}(x, y)]$. This function can evidently have an (x, y) discontinuity greater than or equal to $1/2\sigma$ only at points (x, y) for which (2) holds, that is at a discrete set of points, and this is true for all values of p. Hence the simultaneous integral $\int_\alpha^\beta \int_a^b [S_{s+p}(x, y) - S(x, y)]\, dxdy$ exists. Moreover the value of this integral is less than or at most equal to $(x - a)(y - a)\sigma + 2G\varepsilon$. Since by hypothesis the simultaneous integrals $\int_\alpha^\beta \int_a^b S_{s+p}(x, y)\, dxdy$ and $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ also exist for any (x, y) of the region, we have

$$\int_\alpha^\beta \int_a^b S_{s+p}(x, y)\, dxdy - \int_\alpha^\beta \int_a^b S(x, y)\, dxdy = \int_\alpha^\beta \int_a^b [S_{s+p}(x, y) \\ - S(x, y)]\, dxdy < (x - a)(y - a)\sigma + 2G\varepsilon < \varepsilon'.$$

Hence we obtain in the limit $\lim_{n=\infty} \int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy = \int_\alpha^\beta \int_a^b S(x, y)\, dxdy$, which was to be proved.

Before taking up the case where G has no upper finite limit, we must first prove a couple of lemmas. These theorems were first proved by Scheeffer* for a function of single variable.

Lemma 1.—If $F(x, y)$ and $f(x, y)$ are two functions of x and y which are continuous in (x, y) together throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

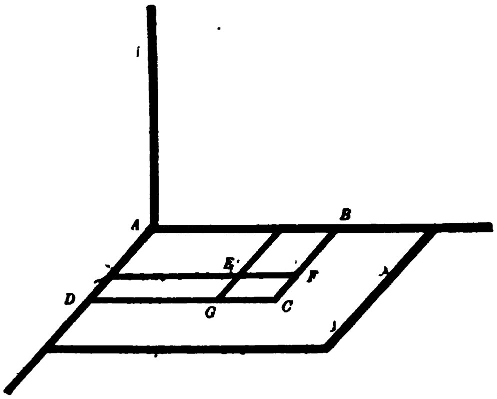

FIG. 14.

and if for every point (x, y) of this region two corresponding difference quotients are always equal, that is if

$$\frac{F(x+h, y+k) - F(x, y+k) - F(x+h, y) + F(x, y)}{hk} =$$
$$\frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk},$$

these values being finite, then $F(x, y)$ and $f(x, y)$ can differ at most by a constant quantity.

In order to prove this, we consider the function

$$\psi(x, y) = c\,(x-\alpha)(y-a) + F(x, y) - f(x, y) - [F(\alpha, a) - f(\alpha, a)]. \tag{1}$$

Let $DF(x, y)$, $Df(x, y)$, and $D\psi(x, y)$, be the corresponding difference quotients and let c be any positive quantity. Then $\psi(x, y)$ can never be negative in the region (A). In order to prove this we assume the contrary and suppose that $\psi(x, y)$ can be negative at some point $(\bar{x}, \bar{y})$ of the given region. Let this be the point C and consider now the rectangle $A(\alpha, a)$, $B(\bar{x}, a)$, $C(\bar{x}, \bar{y})$, $D(\alpha, \bar{y})$, see figure 14.

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* Cf. L. Scheeffer: Acta Mathematica, Vol. 5 (1884), pages 84 and 183.

Within this rectangle the points (x', y') at which (x, y) is not negative are such that the absolute value of $\sqrt{[x'^2 + y'^2]}$ has an upper limit which is finite. Suppose that this occurs at the point $(\bar{x}', \bar{y}') = E$. Since $\psi(x, y)$ is continuous at every point of region (A) in (x, y) together, it follows that $\psi(\bar{x}', \bar{y}') = 0$. But $\bar{x}' < \bar{x}$ and $\bar{y}' < \bar{y}$. Moreover we have for every point of the rectangle *EFCG*

$$\psi(x, y) - \psi(\bar{x}', \bar{y}') = 0.$$

From this follows that $D\psi(\bar{x}', \bar{y}') = 0$. On the other hand, $Df(x, y)$ and $D\psi(x, y)$ are both finite, and hence we have from (1) that $D\psi(x, y) = c$ for every point (x, y) of region (A). This last statement is surely true for

$$D\psi(x, y) = Dc(x - a)\,(y - a) + DF(x, y) - Df(x, y) = Dc(x - a)\,(y - a).$$

But $Dc(x - a)\,(y - a) = c$. We are thus led to a contradiction, and since this holds for every positive value of c, it follows that (x, y) can never be negative. Hence

$$F(x, y) - f(x, y) - [F(a, a) - f(a, a)]$$

cannot ever be negative. In a like manner we can prove that this difference cannot ever be positive and hence it must be zero. This proves the lemma.

This lemma may be extended as follows:

Lemma 2.—If the conditions of lemma 1 hold, with the exception that the two difference quotients are not equal to each other at an enumerable set of points, that is, are not everywhere dense in any sub-region of region (A), then the two given functions can differ at most by a constant quantity.

Let L be a set of exceptional points, and suppose that L is not everywhere dense in any sub-region of region (A). We then have a set D of sub-regions d which do not contain any of the exceptional points, and we can thus determine a closed set of points C, with respect to which L is everywhere dense. In every sub-region d the preceding theorem holds, so that for every sub-region d' contained wholly within d the difference $F(x, y) - f(x, y)$ is constant. But this difference is continuous in (x, y) together and hence it follows that it is constant also in the sub-region d. If this is so, then $F(x, y) - f(x, y)$ is either a constant or it is "streckenweise" discontinuous. (See Schoenfliess: Punktmannigfaltigkeiten, p. 166). If L is enumerable, then the above difference is constant, while if L has the "Mächtigkeit" of the continuum then this difference is "streckenweise" discontinuous. This proves the lemma.

We are now ready to take up our final theorem.

Theorem 3.—If we have

$$S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \dots + u_n(x, y),$$

where $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$ for every (x, y) of

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

and if $S(x, y)$ is completely defined and is always less in absolute value than some finite quantity, and if both the simultaneous integrals $\int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy$ and $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ exist, then the sufficient condition that

$$\lim_{n=\infty} \int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy = \int_\alpha^\beta \int_a^b S(x, y)\, dxdy$$

where $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ is finite, is that the set of points (x', y') for which we cannot assign a circle of radius $r > 0$ and an integer m, such that for every point (x, y) of this circle and for every $n > m$,

$$|S_n(x, y)| < G \qquad (G \text{ finite})$$

is enumerable.

In this case we have in the plane $1/n = z$ certain points (x', y') in whose neighborhood $S_n(x, y)$ does not have a finite upper limit. About such a point as center we can therefore construct a circle of radius $r > 0$, and at the same time find a plane $z = 1/m_s$, such that for every point (x, y) of this circle and for every value of $p = 1, 2, 3, \dots,$

$$|S_{m_s+p}(x, y)| > G. \qquad (G \text{ finite})$$

This set of points (x', y') must evidently be a closed set, for it is a set for which the limit of the oscillation k is greater than some fixed finite number. We have now by hypothesis

$$\int_\alpha^x \int_a^y S(x, y)\, dxdy = \theta(x, y)$$

and

$$\lim_{n=\infty} \int_\alpha^x \int_a^y S_n(x, y)\, dxdy = \phi_0(x, y).$$

Now let (x, y) be some point not belonging to the set (x', y'). There exists then about it a circle of radius $r > 0$ for which

$$|S_{m_s+p}(x, y)| < G.$$

Consider now the ratios

$$\frac{\theta(x+h,y+k)-\theta(x+h,y)-\theta(x,y+k)+\theta(x,y)}{hk}$$

and

$$\frac{\phi_0(x+h,\,y+k)-\phi_0(x+h,\,y)-\phi_0(x,\,y+k)+\phi_0(x,\,y)}{hk}$$

where $(x+h,\,y+k)$ is to be *a* point of the above circle. Choose an arbitrarily small $\lambda>0$; we can then find a plane $z=1/m_s$, such that for every value of $p=1,\,2,\,3,\ldots.$

$$\frac{\phi_0(x+h,\,y+k)-\phi_0(x+h,\,y)-\phi_0(x,\,y+k)+\phi_0(x,\,y)}{hk}=$$

$$\frac{\phi_{m_s+p}(x+h,\,y+k)-\phi_{m_s+p}(x+h,\,y)-\phi_{m_s+p}(x,\,y+k)+\phi_{m_s+p}(x,\,y)}{hk}+$$

$$-1<\mu<1,$$

since $\lim\limits_{n=\infty}\phi_{m_s+p}(x,\,y)=\phi_0(x,\,y)$ for every point $(x,\,y)$. But we have by hypothesis

$$\frac{\phi_{m_s+p}(x+h,\,y+k)-\phi_{m_s+p}(x+h,\,y)-\phi_{m_s+p}(x,\,y+k)+\phi_{m_s+p}(x,\,y)}{hk}=$$

$$1/hk\int_x^{x+h}\int_y^{y+k}S(x;y)\,dxdy+\mu\lambda$$

$$-1<\mu<1,$$

where λ is independent of x, y, h, and k. Since now $|S_n(x,\,y)|<G$ at the points in question, it follows from theorem 2 that

$$1/hk\int_x^{x+h}\int_y^{y+k}S_{m_s+p}(x,\,y)\,dxdy=1/hk\int_x^{x+h}\int_y^{y+k}S(x,\,y)\,dxdy+\mu'\frac{\lambda}{h}$$

$$-1<\mu'<1.$$

Hence

$$\frac{\phi_0(x+h,\,y+k)-\phi_0(x+h,\,y)-\phi_0(x,\,y+k)+\phi_0(x,\,y)}{hk}=$$

$$1/hk\int_x^{x+h}\int_y^{y+k}S(x,\,y)\,dxdy+\mu'\frac{\lambda}{h}+\mu\lambda$$

$$=\frac{\theta(x+h,\,y+k)-\theta(x+h,\,y)-\theta(x,\,y+k)+\theta(x,\,y)}{hk}+\lambda\left\{\mu+\frac{\mu'}{h}\right\}.$$

which holds for every x, $x+h$, y, and $y+k$, determined as above. We thus obtain two different quotients which are equal to each other at all points of region (A), excepting at a closed enumerable set of points, and hence by lemma 2 they can differ from each other at most by a constant quantity. Hence

$$\phi_0(x, y) = \theta(x, y) + \text{constant}.$$

But

$$\phi_0(x, y) = \theta(a, y) = 0$$

hence

$$\phi_0(x, y) = \theta(x, y),$$

which was to be proved.

By a proof similar to that just employed, we can show that the theorem holds also when we replace the function $S_n(x, y)$ by the general function $f(x,y,z)$.

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